Program Verification: Lecture 10

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Termination

We need methods to check termination of an equational theory (Σ, E). This means proving that the rewriting relation $\longrightarrow_E$ (or, more generally, $\longrightarrow_{E/B}$ for $(\Sigma, E \cup B)$) is well-founded.

The key observation is that, if we exhibit a well-founded ordering $>$ on terms such that

$$(\clubsuit) \quad t \longrightarrow_E t' \quad \Rightarrow \quad t > t',$$

then we have obviously proved termination, since nontermination of $\longrightarrow_E$ would make the order $>$ non-well-founded.

Reduction Orderings

To show (♣) we need to consider an, infinite number of rewrites $t \longrightarrow_E t'$. We would like to reduce this problem to checking (♣) only for the oriented equations in $\vec{E}$. We need:

Definition: A well-founded ordering $>$ on $\cup_{s \in S} T_\Sigma(V)$ is called a reduction ordering iff it satisfies the following two conditions:

- strict Σ-monotonicity: for each $f \in \Sigma$, whenever $f(t_1, \ldots, t_n), f(t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_n) \in T_\Sigma(V)$ with $t_i > t'_i$, then, $f(t_1, \ldots, t_n) \; > \; f(t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_n)$
- closure under substitutuion: if $t > t'$, then, for any substitution $\theta : V \longrightarrow T_\Sigma(V)$ we have, $t\theta > t'\theta$.

$>$ is called a simplification order if, furthermore, $f(t_1, \ldots, t_n) \; > \; t_i$, $1 \leq i \leq n$.

Reduction Orderings (II)

Theorem 1: For (Σ, E) an equational theory, $\vec{E}$ is terminating iff there exists a reduction order $>$ s.t. $\forall (u = v) \in E$, $u > v$.

Proof: $(\Rightarrow)$ follows from: (i) $\vec{E}$ terminating makes $\longrightarrow_{E}^{+}$ irreflexive and well-founded order, (ii) the Context Lemma implies $\longrightarrow_{E}^{+}$ strictly monotonic, and (iii) the Substitution Lemma implies $\longrightarrow_{E}^{+}$ closed under substitutions.

To see $(\Leftarrow)$, need to prove $t \to_E t' \;\Rightarrow\; t > t'$. By $t \to_{\vec{E}} t'$, there is a position p, equation $(u = v) \in E$, and substitution θ s.t. $t = t[u\theta]_p$, and $t' = t[v\theta]_p$. We reason by induction on the length $|p|$ of p. Base Case: $p = \epsilon$, i.e., $t = u\theta > v\theta = t'$, by $>$ substit. closed. Induction Step: $p = i.q$ has length $n + 1$, with $t = f(w_1, \ldots, w_i, \ldots, w_n)$, $t' = f(w_1, \ldots, w_i', \ldots, w_n)$, with $w_i \to_E w_i'$ at position q. By Ind. Hyp. $w_i > w_i'$, and by strict monotonicity $t > t'$. q.e.d.

Recursive Path Ordering (RPO)

The recursive path ordering (RPO) is based on the idea of giving an ordering on the function symbols in Σ, which is then extended to a reduction ordering on all terms. Since if Σ is finite the number of possible orderings between function symbols in Σ is also finite, checking whether a proof of termination exists this way can be automated.

The intuitive idea that functions that are more complex should be bigger in the ordering (for example: `_*_` $>$ `_+_` $>$ `s` $>$ `0`) tends to work quite well, and can yield a reduction ordering contaning the equations. Furthermore each symbol f in Σ is given a status $\tau(f)$ equal to either: $\tau(f) = lex(\pi)$ (lexicographic), or $\tau(f) = mult$ (multiset). $\tau(f)$ indicates how the arguments of f should be compared in the order.

RPO (II)

Given a finite signature $\Sigma = ((S, <), F, \Sigma)$, plus an ordering $>$ and a status function τ on its symbols F, the recursive path ordering $>_{rpo}$ on $\cup_{s \in S} T_{\Sigma}(V)$ is defined recursively as follows. $u >_{rpo} t$ iff:

$u = f(u_1, \ldots, u_n)$, and either:

1. $u_i \geq_{rpo} t$ for some $1 \leq i \leq n$, or
2. $\mathrm{t} = g(t_1, \ldots, t_m)$, $u >_{rpo} t_j$ for all $1 \leq j \leq m$, and either:
   - $f > g$, or
   - $f = g$ and $\langle u_1, \ldots, u_n \rangle >_{rpo}^{\tau(f)} \langle t_1, \ldots, t_n \rangle$

where the extension of $>_{rpo}$ to an order $>_{rpo}^{\tau(f)}$ on lists of terms is explained below.

RPO (III)

The extension of $>_{rpo}$ to an order $>_{rpo}^{\tau(f)}$ on lists of terms is defined as follows:

- If f has n arguments and $\tau(f) = lex(\pi)$ with π a permutation on n elements, then $\langle u_1, \ldots, u_n \rangle >_{rpo}^{\tau(f)} \langle t_1, \ldots, t_n \rangle$ iff there exists i, $1 \leq i \leq n$ such that for $j < i$ $u_{\pi(j)} = t_{\pi(j)}$, and $u_{\pi(i)} > t_{\pi(i)}$.
- if $\tau(f) = mult$, then $\langle u_1, \ldots, u_n \rangle >_{rpo}^{\tau(f)} \langle t_1, \ldots, t_n \rangle$ iff we have $\{u_1, \ldots, u_n\} >_{rpo}^{mult} \{t_1, \ldots, t_n\}$

where, given any order $>$ on a set A, it extension to an order $>^{mult}$ on the set $Mult(A)$ of multisets on A is the transitive closure of the relation $>_{elt}^{mult}$ defined by $M \cup \{a\} >_{elt}^{mult} M \cup S$ iff $(\forall x \in S)\ a > x$, where S can be $\emptyset$.

RPO (IV)

It can be shown (for detailed proofs see the Terese and Baader-Nipkow books cited later) that any RPO order $>$ on a finite signature Σ is a simplification order. By Theorem 1, we can therefore use it to prove termination of $\vec{E}$, by just checking that $u > v$ for each $(u \to v) \in \vec{E}$.

Consider for example the usual equations for natural number addition: $n + 0 = n$ and $n + s(m) = s(n + m)$. We can prove that they are terminating by using the RPO associated to the ordering $+ > s > 0$ with $\tau(f) = lex(id)$ for each symbol f. Indeed, it is then trivial to check that $n + 0 >_{rpo} n$ and $n + s(m) >_{rpo} s(n + m)$.

Termination Modulo Axioms B

To prove that the rewrite relation $\rightarrow_{E/B}$ is terminating, we need a reduction order that is compatible with the axioms B. That is, if $u > t$, $u =_B u'$ and $t =_B t'$, then we must also have $u' > t'$. This means that $>$ defines also an order on the set, $\cup_{s \in S} T_{\Sigma/B}(X)$. For example, RPO is compatible with commutativity axioms if we specify $\tau(f) = mult$ for each commutative symbol f.

To make *RPO* compatible with associative and commutative symbols it has been generalized to the *AC.RPO* order by a method of flattening *AC* symbols. E.g., for f *AC*, $f(f(a, b), f(c, d))$ flattens to $f(a, b, c, d)$. In his Ph.D. thesis,[a] Albert Rubio has further generalized *AC.RPO* to the $A \vee C.RPO$ order, where some symbols can be associative and/or commutative.

[a] A. Rubio, "Automated Deduction with Constrained Clauses," Ph.D. thesis, Universitat Politècnica de Catalunya, 1994.

Proving Termination with $A \vee C.RPO$

The Maude Termination Assistant (MTA) can prove termination modulo $A \vee C$ axioms using an $A \vee C.RPO$ reduction order.

To prove a functional module `FOO` (preceded by: `set include BOOL off .`) $A \vee C.RPO$-terminating:

1. Choose a number n_f for each $f \in \Sigma$ ($f > g$ iff $n_f > n_g$) using Maude's `metadata` attribute to specify n_f and `lex` in `FOO`.

2. Load functional module `FOO` in Maude; then load `mta.maude`.

3. Give the command `(check-AvCrpo FOO .)` that will check whether each $u = v$ in `FOO` satisfies $u >_{A \vee C.rpo} v$. It will reply: `Module is terminating by AvC-RPO order` or display those $u = v$ in `FOO` not provable with chosen order $>$.

MTA proves module `LIST+MSET` is $AC.RPO$-terminating:

Proving Termination with $A \vee C.RPO$ (II)

```
set include BOOL off .

fmod LIST+MSET is
  sorts Element List MSet .  subsorts Element < List .
  subsorts Element < MSet .
  op a : -> Element [ctor metadata "1"] .
  op b : -> Element [ctor metadata "2"] .
  op c : -> Element [ctor metadata "3"] .
  op nil : -> List [ctor metadata "4"] .
  op _;_  : List List -> List [metadata "5 lex(2 1)"] .
  op _;_  : List Element -> List [ctor metadata "5 lex(2 1)"] .
  op _,_ : MSet MSet -> MSet [ctor assoc comm metadata "4"] .
  op null : -> MSet [ctor metadata "3"] .
  op l2m : List -> MSet [ctor metadata "5"] .
  vars L P Q : List .  var M : MSet .  var E : Element .
  eq L ; (P ; Q) = (L ; P) ; Q .                eq L ; nil = L .
  eq nil ; L = L .      eq M , null = M .       eq l2m(nil) = null .
  eq l2m(E) = E .                               eq l2m(L ; E) = l2m(L) , E .
endfm
```

Polynomial Orderings

Another general method of defining suitable reduction orderings is based on polynomial orderings. In its simplest form we can just use polynomials on several variables whose coefficients are natural numbers. For example,

$$p = 7x_1^3x_2 + 4x_2^2x_3 + 6x_3^2 + 5x_1 + 2x_2 + 11$$

is one such polynomial. Note that a polynomial p whose biggest indexed variable is n (in the above example $n = 3$) defines a function $p_{\mathbb{N}_{\geq k}} : \mathbb{N}^n_{\geq k} \longrightarrow \mathbb{N}_{\geq k}$ (where $k \geq 3$ and $\mathbb{N}_{\geq k} = \{n \in \mathbb{N} \mid n \geq k\}$), just by evaluating the polynomial on a given tuple of numbers, each of them greater or equal to k. For p the polynomial above we have for example, $p_{\mathbb{N}_{\geq k}}(3, 3, 3) = 383$.

Polynomial Orderings (II)

Note also that we can order the set $[\mathbb{N}^n_{\geq k} \rightarrow \mathbb{N}_{\geq k}]$ of functions from $\mathbb{N}^n_{\geq k}$ to $\mathbb{N}_{\geq k}$ by defining $f > g$ iff for each $(a_1, \ldots a_n) \in \mathbb{N}^n_{\geq k}$ $f(a_1, \ldots a_n) > g(a_1, \ldots a_n)$. This order on functions is well-founded, since if we have an infinite descending chain of functions

$$f_1 > f_2 > \ldots f_n > \ldots$$

by choosing any $(a_1, \ldots a_n) \in \mathbb{N}^n_{\geq k}$ we would get a descending chain of positive numbers

$$f_1(a_1, \ldots a_n) > f_2(a_1, \ldots a_n) > \ldots f_n(a_1, \ldots a_n) > \ldots$$

which is impossible.

Polynomial Orderings (III)

The method of polynomial orderings then consists in assigning to each function symbol $f : s_1 \ldots s_n \longrightarrow s$ in Σ a polynomial p_f involving exactly the variables $x_1, \ldots, x_n$ (all of them, and only them must appear in p_f). If f is subsort overloaded, we assign the same p_f to all such overloadings. Also, to each constant symbol b we likewise associate a positive number $p_b \in \mathbb{N}_{\geq k}$.

Suppose that in our set E of equations we have used variables $Y = vars(E)$ (such variables need not be numbered at all). Then our assignment of a polynomial p_f on variables $x_1, \ldots, x_n$ to each function symbol of n arguments, and a number p_a to each constant a extends to a function:

Polynomial Orderings (IV)

$$p\_ : T_{\Sigma^u(Y)} \longrightarrow \mathbb{N}[Y]$$

where $\Sigma^u(Y)$ is the unsorted version of $\Sigma(Y)$, $\mathbb{N}[Y]$ denotes the polynomials with natural number coefficients in the variables Y, and where $p\_$ is defined inductively as follows:

- $p_b = p_b$
- $p_y = y$ for each $y \in Y$
- $p_{f(t_1,\ldots,t_n)} = p_f\{x_1 \mapsto p_{t_1}, \ldots, x_n \mapsto p_{t_n}\}$

Polynomial Orderings (V)

Note that the the polynomial interpretation p induces a well-founded ordering $>_p$ on the terms of $T_{\Sigma(Y)}$ as follows:

$$t >_p t' \quad \Leftrightarrow \quad p_{t_{\mathbb{N}_{\geq k}}} > p_{t'_{\mathbf{N}_{\geq k}}}$$

where if $Y = vars(E)$ and $|Y| = m$, linearly ordering Y we interpret $p_{t_{\mathbb{N}_{\geq k}}}$ and $p_{t'_{\mathbf{N}_{\geq k}}}$ as functions in $[\mathbb{N}^m_{\geq k} \to \mathbb{N}_{\geq k}]$. The relation $>_p$ is clearly an irreflexive and transitive relation on terms in $T_{\Sigma(X)} \subseteq T_{\Sigma^u(X)}$, therefore a strict ordering, and is clearly well-founded, because otherwise we would have an infinite descending chain of polynomial functions in $[\mathbb{N}^m_{\geq k} \to \mathbb{N}_{\geq k}]$, which is impossible.

Polynomial Orderings (VI)

We now need to check that this ordering is furthermore: (i) strictly Σ-monotonic, and (ii) closed under substitution. Condition (i) follows from $+$ and $*$ strictly monotonic on $\mathbb{N}_{\geq k}$, plus each function symbol $f : s_1 \ldots s_n \longrightarrow s$ in Σ the polynomial p_f involving exactly the variables $x_1, \ldots x_n$ (p_f does not drop any variables and all coefficients are non-zero). Therefore, $p_{f_{\mathbb{N}_{\geq k}}}$, viewed as a function of n arguments, is strictly monotonic in each of its arguments. Condition (ii) follows from the following general property of the $p\_$ function, left as an exercise, (where $vars(t) = \{y_1, \ldots, y_n\}$):

$$p_{(t\{y_1 \mapsto u_1, \ldots, y_n \mapsto u_n\})} = p_t\{y_1 \mapsto p_{u_1}, \ldots, y_n \mapsto p_{u_n}\}.$$

This then easily yields that if $t >_p t'$ then $t\{y_1 \mapsto u_1, \ldots, y_n \mapsto u_n\} >_p t'\{y_1 \mapsto u_1, \ldots, y_n \mapsto u_n\}$, as desired.

Polynomial Orderings (VII)

Therefore, polynomial interpretations of this kind define reduction orders and can be used to prove termination. One can also easily show them to be simplification orders. Consider for example the single equation $f(g(x)) = g(f(x))$ in an unsorted signature having also a constant a. Is this equation terminating? We can prove that it is so by, e.g., the following polynomial interpretation:

- $p_f = {x_1}^3$
- $p_g = 2x_1$
- $p_a = 3$

since we have the following strict inequality of functions: $((2x)^3)_{\mathbb{N}_{\geq k}} > (2(x^3))_{\mathbb{N}_{\geq k}}$, showing that $f(g(x)) >_p g(f(x))$.

Polynomial Termination Modulo Axioms

Some polynomial interpretations are compatible with certain axioms. For example, a symmetric polynomial, i.e., such that $p(x, y) = p(y, x)$ is compatible with commutativity and can therefore be used to interpret a commutative symbol. For example, $2x + 2y$ is symmetric. Similarly, a polynomial $p(x, y)$ which is symmetric $(p(x, y) = p(y, x))$ and furthermore satisfies the associativity equation $p(x, p(y, z)) = p(p(x, y), z)$ can be used to interpret an associative-commutative symbol. As shown by Bencheriffa and Lescanne the polynomials satisfying associativity and commutativity axioms have a simple characterization: they must be of the form $a\,xy + b(x + y) + c$ with $ac + b - b^2 = 0$.

Proving Polynomial Termination with MTA

The MTA tool can be used to prove polynomial termination of a module `FOO` using linear polynomials. That is, we associate to each n-argument operator $f \in \Sigma$ a linear polynomial of the form:

$$p_f = a_1 x_1 + \ldots + a_n x_n + a_{n+1}$$

where $a_i \neq 0$ for $1 \leq i \leq n$. For constants $c \in \Sigma$ we require $p_c = a_1 \geq 3$.

Using the `metadata` attribute, we express each p_f as the string "$a_1 \ldots a_{n+1}$".

To prove polynomial termination we: (1) load `FOO` into Maude with `metadata` annotations; then load `mta.maude`; then (2) give the command: `(check-poly FOO .)` MTA then replies with either `Module is terminating by polynomial order` or the list of equations failing the given order. Let us see an example:

Proving Polynomial Termination with MTA (II)

```
set include BOOL off .

fmod LIST+MSET is
  sorts Element List MSet .    subsorts Element < List .
  subsorts Element < MSet .
  op a    : -> Element [ctor metadata "3"] .
  op b    : -> Element [ctor metadata "3"] .
  op c    : -> Element [ctor metadata "3"] .
  op nil  : -> List [ctor metadata "2"] .
  op _;_  : List List -> List [metadata "2 1 1"] .
  op _;_  : Element List -> List [ctor metadata "2 1 1"] .
  op _,_  : MSet MSet -> MSet [ctor assoc comm metadata "1 1 1"] .
  op null : -> MSet [ctor            metadata "2"] .
  op l2m  : List -> MSet [ctor            metadata "1 1"] .
  vars L P Q : List .  var M : MSet .  var E : Element .
  eq (L ; P) ; Q = L ; (P ; Q) .                eq L ; nil = L .
  eq nil ; L = L .      eq M , null = M .      eq l2m(nil) = null .
  eq l2m(E) = E .                               eq l2m(E ; L) = E , l2m(L) .
endfm
```

Proving Polynomial Termination with MTA (III)

For an `assoc comm` (or `assoc comm id:`) symbol f, recall that the corresponding polynomial p_f must itself be `assoc comm` and therefore must have the form: $axy + b(x + y) + c$ with $ac + b - b^2 = 0$. But since in MTA p_f must be linear, this forces $a = 0$ and $b = b^2$. Therefore, $p_f = 1x + 1y + c$. That is why we have declared:

```
op _,_  : MSet MSet -> MSet [ctor assoc comm metadata "1 1 1"] .
```

Note that if f is only `assoc` (or `assoc id:`) it is OK for p_f to be `assoc comm`, since in particular p_f is `assoc`. Therefore, for an `assoc` symbol f we must also choose $p_f = 1x + 1y + c$.

Note: We do not need to worry about p_f satisfying `id:` axioms: MTT automatically generates a semantically equivalent module where `id:` axioms become rules, so p_f need only be `assoc comm`.

The MTT Tool

The Maude Termination Tool (MTT) is a tool that can be used to prove the operational termination of Maude functional modules.

Functions in such modules may be declared with axioms like associativity and commutativity; and also with evaluation strategies (see the Maude Manual, Section 4.4.7), indicating what arguments of a function symbol should be evaluated before applying equations for that symbol. For example, in an `if_then_else_fi` the first agument should be evaluated before equations for it are applied; and in a "lazy list cons" `_;_` the first argument is evaluated, but not the second.

The MTT Tool (II)

Features such as sorts, subsorts, and evaluation strategies may be essential for the termination of a Maude module. That is, ignoring them may result in a nonterminating module.

To preserve these features somehow, while still allowing using standard termination backend tools, the MTT implements the transformations of (Σ, E) first into an unsorted conditional theory (Σ°, E°), and then (Σ°, E°) is transformed into an unsorted unconditional theory $(\Sigma^\bullet, E^\bullet)$.

If the module declares evaluation strategies, they are also transformed; but at the end evaluation strategies can either be used directly by a termination tool like Mu-Term, or a further theory transformation can eliminate such strategies.

The MTT Tool (III)

The course web page indicates how the MTT is part of Maude's Formal Environment (MFE).

Once a Maude module has been entered into the MTT (the module should not import any built-in modules like, e.g., `NAT`), the user can perform the theory transformation $(\Sigma, E) \mapsto (\Sigma^{\bullet}, E^{\bullet})$ in one of three increasingly simpler modes: (1) Complete; (2) No Kinds; and (3) No Sorts. In case (2) kinds are ignored; and in case (3) both kinds and sorts are ignored. There is a tradeoff between simplicity of the transformation and its tightness. Sometimes a simpler transformation works better, and sometimes a more complete one is essentially needed.

The MTT Tool (IV)

The choice of transformation can be made by clicking the appropriate buttons (a screenshot will show this). But one also needs to choose which backend termination tool for unsorted and uncondional specifications will be used. One among the CiME, MU-TERM, and AProVE termination tools can be chosen.

Then one can click on the `Check` bar to check the specification with the chosen tool. Some of these tools offer choices for different settings. So, we can try to prove termination using three different transformation variants, and then with one of three backend tools, sometimes customizing the particular tool choices. This maximizes the chances of obtaining a successful termination proof.

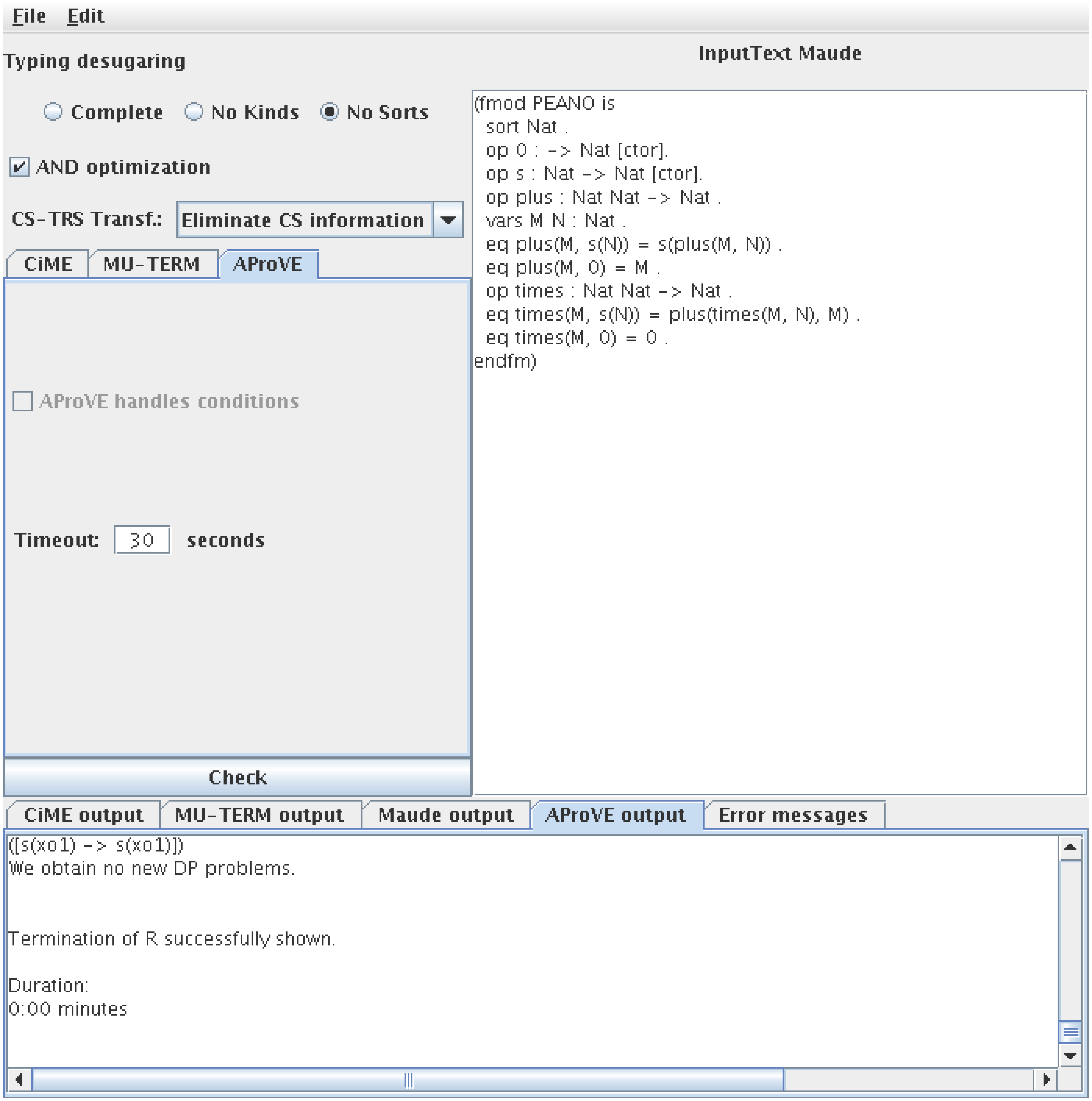
The MTT Tool (V)

What the MTT tool then demonstrates is that the original Maude functional module is terminating. The correctness of such a proof is based on:

- the correctness of the theory transformations[a] and
- the correctness of the chosen tools, that sometimes output a justification of how they proved termination.

A screeshot of a tool interaction is given in the next page.

[a] See F. Durán, S. Lucas an J. Meseguer, "Methods for Proving Termination of Rewriting-based Programming Languages by Transformation." Electron. Notes Theor. Comput. Sci. 248: 93-113 (2009)


File Edit
Typing desugaring
InputText Maude
Complete
No Kinds
No Sorts
AND optimization
CS-TRS Transf.:
Eliminate CS information
CiME
MU-TERM
AProVE
AProVE handles conditions
Timeout:
30
seconds
Check
(fmod PEANO is
sort Nat .
op 0 : -> Nat [ctor].
op s : Nat -> Nat [ctor].
op plus : Nat Nat -> Nat .
vars M N : Nat .
eq plus(M, s(N)) = s(plus(M, N)) .
eq plus(M, 0) = M .
op times : Nat Nat -> Nat .
eq times(M, s(N)) = plus(times(M, N), M) .
eq times(M, 0) = 0 .
endfm)
CiME output
MU-TERM output
Maude output
AProVE output
Error messages
([s(xo1) -> s(xo1)])
We obtain no new DP problems.
Termination of R successfully shown.
Duration:
0:00 minutes

Termination is Undecidable

All the termination tools try to prove that a set of equations E, conditional or unconditional, is terminating by applying different proof methods; for example by trying to see if particular orderings can be used to prove the equations terminating.

But these termination proof methods are not decision procedures: in general termination of a set of equations (even if they are unconditional) is undecidable. This is so because a Turing machine can be specified as a term rewriting system (TRS) (see Ölveczky's, and Baader and Nipkow's books in next slide) and the halting (=termination) problem is undecidable for Turing machines. However, there are some TRS classes (e.g., ground TRSs) for which termination is decidable (see Baader and Nipkow's book in next slide).

Where to Go from Here

Besides RPO and polynomials, other orderings and termination methods can be used to prove termination. Good sources include:

TeReSe, "Term Rewriting Systems," Cambridge U. P., 2003.

Baader and Nipkow, "Term Rewriting and All That", Cambridge U.P., 1998.

N. Dershowitz and J.-P. Jouannaud, "Rewrite Systems," in J. van Leeuwen, ed., "Handbook of Theoretical Computer Science," Elsevier, 1990.

E. Ohlebusch, "Advanced Topics in Term Rewriting Systems," Springer Verlag, 2002.

P. Ölveczky, "Designing Reliable Distributed Systems," Springer Verlag, 2017.

Exercises

Ex.10.1 Prove that if $>$ is well-founded, then its multiset extension $>^{mult}$ is also well-founded.

Ex.10.2. Prove that if $>$ is a well-founded relation on a set A of elements, then, given two multisets, M, N with elements in A, $M >^{mult} N$ holds iff:

- M and N can be decomposed as: $M = S \cup \{a_1, \ldots, a_k\}$, $k \geq 1$, $a_i \in A$, and $M = S \cup U_1 \cup \ldots \cup U_k$ (where M and some of the U_i could be $\emptyset$),
- $\{a_1, \ldots, a_k\} \cap (U_1 \cup \ldots \cup U_k) = \emptyset$, and
- $\forall x \in U_i,\ a_i > x,\ \ 1 \leq i \leq k.$

Ex.10.3. Prove that any RPO order is a simplification order.