Lecture 6: Non Context Free Languages

Date: September 7, 2023.

Pumping Lemma: For a CFL L, there is $p \geq 0$ (pumping length) such that for every $z \in L$, if $|z| \geq p$ then there are strings u, v, w, x, y such that $z = uvwxy$ such that

1. $vx \neq \epsilon$,
2. $|vwx| \leq p$, and
3. for every $i \geq 0$, $uv^iwx^iy \in L$.

L is CFL $\Rightarrow \exists p$

PL $\left\{ (\forall z.\ |z| \geq p \wedge z \in L \rightarrow (\exists u, v, w, x, y.\ (z = uvwxy \wedge |vx| \neq 0 \wedge |vwx| \leq p \wedge \forall i\ uv^iwx^iy \in L))) \right.$

If L does not satisfy PL then L is not CFL.

Defender		Challenger
p	$\xrightarrow{p}$	$z \in L$, $|z| \geq p$
	$\xleftarrow{z}$	
u, v, w, x, y $z = uvwxy$	$\xrightarrow{u,v,w,x,y}$	picks i

If Defender has winning strategy then L satisfies PL

If Challenger has winning strategy then L does not satisfy PL.

Proposition 1. *The language $L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\} \subseteq \{a, b, c\}^*$ is not context-free.*

We will show that L_{anbncn} does not satisfy PL.

$\Rightarrow$ Show that the Challenger has a winning strategy

Let p is pumping length.

Take $z = a^p b^p c^p \in L$, $|z| = 3p \geq p$.

Let u, v, w, x, y s.t. (a) $z = uvwxy$, (b) $vx \neq \epsilon$, (c) $|vwx| \leq p$.

Claim: $uv^0wx^0y \notin L_{anbncn}$

Proof: $|vwx| \leq p \Rightarrow vwx$ doesn't contain both a & c.

WLOG vwx does not contain any c's.

uv^0wx^0y — has p c's but has $< p$ a's or b's.

$\notin L_{anbncn}$.

Problem 1. Show that $L_{a=c\wedge b=d} = \{a^i b^j c^i d^j \mid i, j \geq 0\}$ is not context-free.

Challenger has a winning strategy.

Let p be the pumping length.

pick ~~any~~ $i = p-1$, $j = 1$ and take $z = a^{p-1} b c^{p-1} d$. $z = a^p b^{2p} c^p d^{2p}$

Let u, v, w, x, y s.t. $z = uvwxy$, $vx \neq \epsilon$, $|vwx| \leq p$.

Claim: $uv^0wx^0y \notin L_{a=c\wedge b=d}$. [Not true because $u = a^{p-2}$, $v = a$, $w = b$, $x = \epsilon$, $y = c^{p-2}d$.

$z = a^p b^{2p} c^p d^{2p}$

Let: u, v, w, x, y be s.t. $z = uvwxy$, $vx \neq \epsilon$, $|vwx| \leq p$.

Claim: $uv^0wx^0y \notin L_{a=c\wedge b=d}$.

$|vwx| \leq p \Rightarrow vwx$ cannot contain both a & c and cannot contain both b & d.

WLOG, assume vx contains a but not c.

Then uv^0wx^0y has fewer a's than c's.

Problem 2. Show that $A = \{ww \mid w \in \{0, 1\}^*\}$ is not context-free.

Let p be the pumping length. [$z = (0+1)^p(0+1)^p$

$z = 0^p 1^{2p} 0^p 1^{2p}$] $z = 0^p 1^p 0^p 1^p$.

Let u, v, w, x, y s.t $z = uvwxy$, $vw \neq \epsilon$, $|vwx| \leq p$.

Claim: $uv^0wx^0y \notin A$.

Proof: $|vwx| \leq p$, vwx cannot contain both substring of 1s & cannot contain both substrings of 0's

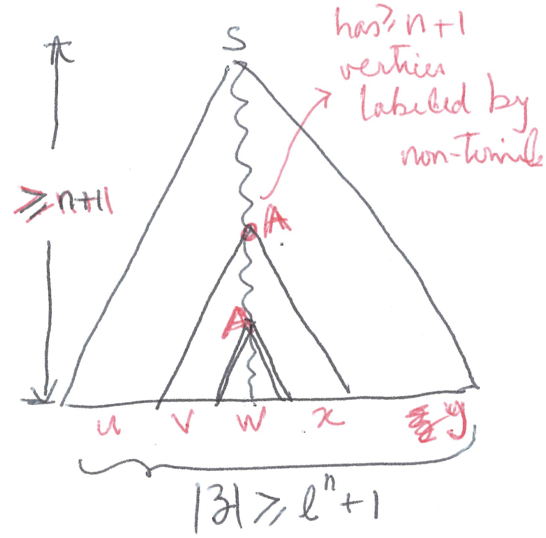
$uv^0wx^0y = 0^i 1^j 0^p 1^p$

$0\cdots0\underbrace{1\cdots1\,0}_{vwx}\cdots01\cdots1$

Pumping Lemma: For a CFL L, there is $p \geq 0$ (pumping length) such that for every $z \in L$, if $|z| \geq p$ then there are strings u, v, w, x, y such that $z = uvwxy$ such that

1. $vx \neq \epsilon$,
2. $|vwx| \leq p$, and
3. for every $i \geq 0$, $uv^iwx^iy \in L$.

Proof: Since L is CFL. $\exists G$. $L = L(G)$



Let n is # nonterminals in G.

Let l be the length of the longest rule

Let $p = l^n + 1$.

Consider $z \in L$ s.t $|z| \geq p$.

Let T be a parse with root S for z that has the fewest vertices.

- T is an l-ary tree.
- T has height $\geq n+1$
- Longest path in T has some non-terminal repeated. (say A)

$\exists$ derivation $S \xrightarrow[G]{*} uAy \xrightarrow[G]{*} uvAxy \xrightarrow[G]{} uvwxy$.

$A \xrightarrow[G]{*} vAx$ $\qquad$ $A \xrightarrow[G]{*} w$

$S \xrightarrow[G]{*} uAy \xrightarrow[G]{*} uv^0wx^0y$ $\qquad$ $S \xrightarrow[G]{*} uAy \xrightarrow[G]{*} uvAxy \xrightarrow[G]{*} uvvAxxy$

$\cdots \xrightarrow[G]{*} uv^iAx^iy$.

$|vwx| \leq p \Rightarrow$ higher A node is at height $\leq n+1$

$vx \neq \epsilon$. because if $vx = \epsilon$ then there is parse tree with fewer vertices that produces z.