Lecture 3: Kleene's Theorem and Closure Properties

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Theorem 1. *For any two DFAs, M_1 and M_2, there is a DFA M such that $\mathbf{L}(M) = \mathbf{L}(M_1) \cup \mathbf{L}(M_2)$.*

$M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ $\quad L_1 = L(M_1)$

$M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ $\quad L_2 = L(M_2)$

$L_1 \cup L_2$: Given input w. Answer Yes if $w \in L_1 \cup L_2$ if $w \in L_1$ or $w \in L_2$. if w is accepted by M_1 or w is accepted by M_2.

Algo for $L_1 \cup L_2$: Input w.

Run M_1 on w $\Big\}$ Answer "yes" if either one does.
Run M_2 on w

$$M_1 \times M_2 = (Q_1 \times Q_2, \Sigma, \delta, (s_1, s_2), F)$$

$$\delta((p_1, p_2), a) = (\delta_1(p_1, a), \delta_2(p_2, a))$$

$$F = F_1 \times F_2 \longrightarrow L(M_1 \times M_2) = L(M_1) \cap L(M_2)$$

$$F = (F_1 \times Q_2) \cup (Q_1 \times F_2) = \{(p_1, p_2) \mid p_1 \in F_1 \text{ or } p_2 \in F_2\} \rightarrow L(M_1 \times M_2) = L(M_1) \cup L(M_2)$$

NFA with ϵ-transitions is a NFA that can take change its state *without reading an input symbol.* Transitions taken without reading an input symbol are called "ϵ-transitions". Formally, it is a tuple $N = (Q, \Sigma, \Delta, S, F)$ where Q, Σ, S, and F are the states, input alphabet, set of start states and final states (as for NFAs) and the transition function $\Delta : Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q$.

A **run** of N on input x is a sequence $a_1 a_2 \ldots a_n$ ~~a sequence $a_1, a_2, \ldots a_n$ and~~ a sequence of states $q_0, q_1, \ldots q_n$ such that (a) $a_i \in \Sigma \cup \{\epsilon\}$ for each i, (b) $x = a_1 a_2 \cdots a_n$, (c) $q_0 \in S$, and (d) $q_{i+1} \in \Delta(q_i, a_i)$ for every $i \geq 0$. An accepting run is one where $q_n \in F$. And an input x is **accepted** if N has **some** accepting run on x. $\mathbf{L}(N)$ is the collection of all strings accepted by N.

Theorem 2. *For every NFA with ϵ-transitions N, there is an NFA M such that $\mathbf{L}(N) = \mathbf{L}(M)$.*

Theorem 3. *For any two NFAs M_1, M_2*

1. *there is an NFA with ϵ-transitions N such that $\mathbf{L}(N) = \mathbf{L}(M_1)\mathbf{L}(M_2)$, and*
2. *there is an NFA with ϵ-transitions N such that $\mathbf{L}(N) = (\mathbf{L}(M_1))^*$.*

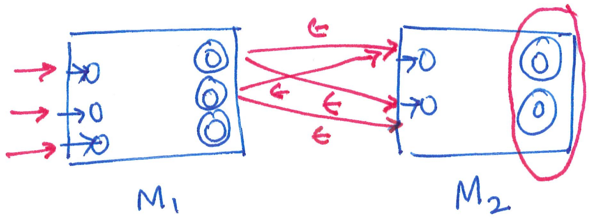
$M_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1)$

$M_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2)$

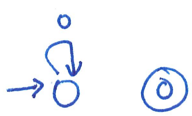
$L_1 = L(M_1)$, $L_2(M_2)$

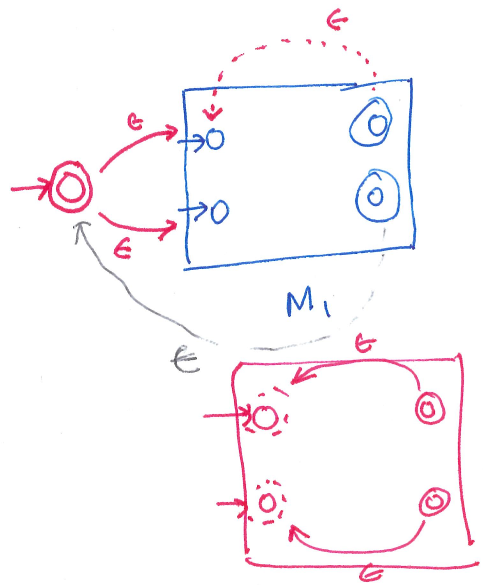
L_1L_2: Input w
$\exists u, v.$ $u \in L_1$ & $v \in L_2$
$w = uv$

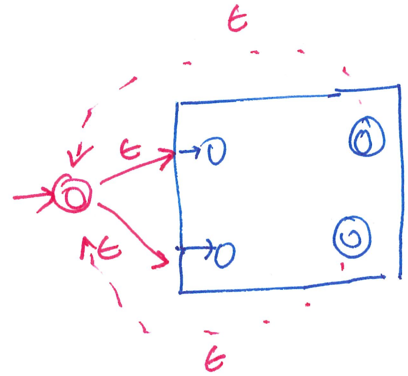
L_1^*: Input w.
$\exists u_1, u_2 \ldots u_n.$ $u_1 u_2 \ldots u_n = w$
$\forall u_i \in L_1$



Guess where to divide the string into 2 parts and check that the first part $\in L_1$ & secon part $\in L_2$.







Theorem 4 (Kleene). *A language is regular if and only if it is recognized by a DFA (NFA, UFA, 2DFA).*

($\Rightarrow$) Regular languages are inductively built from $\boxed{\phi, \{\epsilon\}, \{a\}}$ using concatenation, union and Kleene Closure.
$\exists$ NFA with ϵ-transitions that solves any regular.

($\Leftarrow$) Proof Lecture 9 of the book.

Homomorphism: A function $h : \Sigma^* \to \Gamma^*$ is a *homomorphism* if and only if $h(\epsilon) = \epsilon$, and for every $x, y \in \Sigma^*$, $h(xy) = h(x)h(y)$. $\longrightarrow$

Proposition 5. *Consider homomorphisms $h_1, h_2 : \Sigma^* \to \Gamma^*$. $h_1 = h_2$ if and only if for all $a \in \Sigma$, $h_1(a) = h_2(a)$.*

($\Rightarrow$) Obvious

($\Leftarrow$) Assume $h_1(a) = h_2(a)$ $\forall a \in \Sigma$. Goal: Prove $\forall w \in \Sigma^*$ $h_1(w) = h_2(w)$

By induction on $|w|$.

$h_1(u \cdot a) = h_1(u) \cdot h_1(a) = h_2(u) h_2(a) = h_2(ua)$

Homomorphic and Inverse Homomorphic Images: Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism, $A \subseteq \Sigma^*$, and $B \subseteq \Gamma^*$. Then,

$$h(A) = \{h(w) \mid w \in A\}$$
$$h^{-1}(B) = \{w \mid h(w) \in B\}$$

Example: Let $\Sigma = \{0, 1\}$ and $\overline{\Sigma} = \{\overline{0}, \overline{1}\}$. Consider homomorphisms unbar and rembar defined as

$\{0,1,\overline{0},\overline{1}\}^* \to \{0,1\}^*$

unbar(0) = 0	unbar(1) = 1	unbar($\overline{0}$) = 0	unbar($\overline{1}$) = 1
rembar(0) = 0	rembar(1) = 1	rembar($\overline{0}$) = ϵ	rembar($\overline{1}$) = ϵ

1. unbar($\overline{1}\overline{0}10$) = 1010   rembar($\overline{1}\overline{0}10$) = 10
2. unbar($\Sigma^*\overline{1}\overline{1}\Sigma^*$) = $\Sigma^* 11 \Sigma^*$ rembar($\Sigma^*\overline{1}\overline{1}\Sigma^*$) = Σ^*
3. unbar^{-1}($\{10\}$) = $\{\overline{1}\overline{0}, \overline{1}0, 1\overline{0}, 10\}$
4. rembar^{-1}($\{10\}$) = $\overline{\Sigma}^* 1 \overline{\Sigma}^* 0 \overline{\Sigma}^*$

Theorem 6. *Regular languages are closed under both homomorphic and inverse homomorphic images.*

If L is regular and h is homomorphism then $h(L)$ & $h^{-1}(L)$ are regular.