CS 475: Formal Models of Computation

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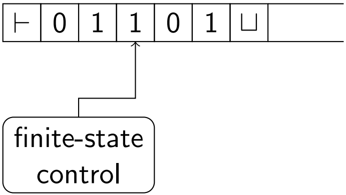
Turing Machine



- A semi-infinite tape with $\vdash$ in leftmost cell
- Initially input stored on tape, with rest of the cell $\sqcup$
- In one step, machine reads symbol under head, and based on current state, changes state, writes a new symbol in cell, and moves head either L or R.

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- $r \in Q$ $(r \neq t)$ is the unique *rejecting state*,
- $\delta : (Q \setminus \{t, r\}) \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function that never overwrites $\vdash$.

Configuration, and One step

- A configuration of a TM must describe the state, contents of the tape, and position of the head. Thus, $\alpha \in Q \times \{y \sqcup^{\omega} \mid y \in \Gamma^*\} \times \mathbb{N}$.

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- The starting configuration on input x is $(s, \vdash x\sqcup^\omega, 0)$
- For a tape $z = y\sqcup^\omega$ $(y \in \Gamma^*)$, $s_b^n(z)$ is the string obtained from z by substituting b for z_n. The next configuration relation is given by

$$\delta(p, z_i) = (q, b, \mathrm{L}) \Rightarrow (p, z, i) \xrightarrow[M]{1} (q, s_b^i(z), i-1),$$
$$\delta(p, z_i) = (q, b, \mathrm{R}) \Rightarrow (p, z, i) \xrightarrow[M]{1} (q, s_b^i(z), i+1).$$

Acceptance, Rejection, and Halting

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- M is total if it halts on all inputs x

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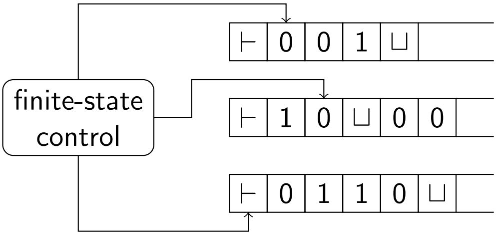
Multi-Tape Turing Machine



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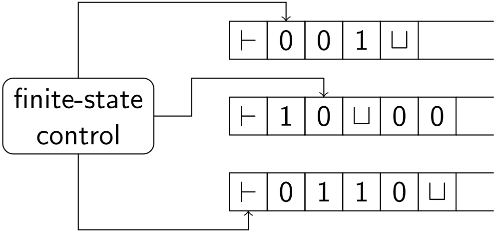

- Input on Tape 1, with left endmarker in cell 0

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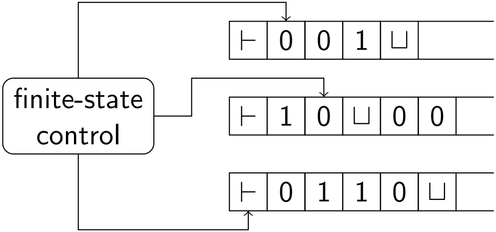

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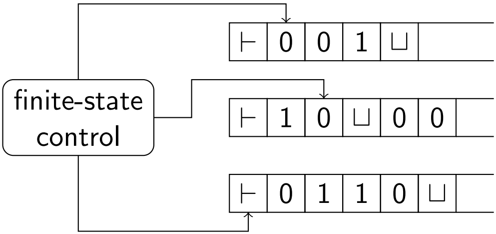

- Input on Tape 1, with left endmarker in cell 0
- Initially all heads scanning cell 0, and tapes 2 to k blank except for the left endmarker
- In one step: Read symbols under each of the k-heads, and depending on the current control state, write new symbols on the tapes, move the each tape head (possibly in different directions), and change state.

Expressive Power of multi-tape TM

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Theorem

For any k-tape Turing Machine M, there is a single tape TM single(M) *such that* $\mathrm{L}(\mathrm{single}(M)) = \mathrm{L}(M)$.

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- How do we store k-tapes in one?

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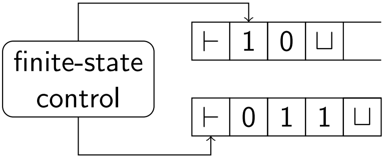
- How do we simulate the movement of k independent heads?

Storing Multiple Tapes



Multi-tape TM M

Store in cell $i+1$ contents of cell i of all tapes.

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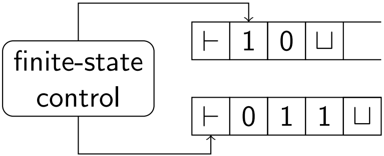

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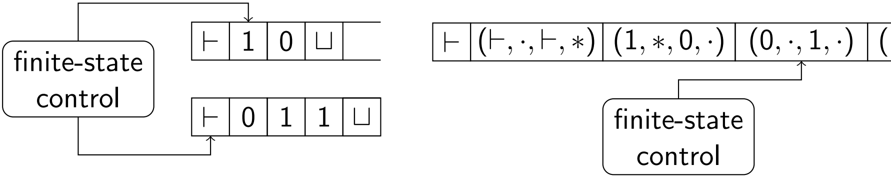

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1-tape equivalent single(M)

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- Once again, scan the tape, change all relevant contents, "move" heads (i.e., move $*$s), and change state.

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Formal construction in notes.

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- Idea 1: $\mathrm{det}(N)$ tries to keep track of all possible "configurations" that N could possibly be after each step. Works for DFA simulation of NFA but not convenient here.
- Idea 2: $\mathrm{det}(N)$ will simulate N on each possible sequence of computation steps that N may try in each step.

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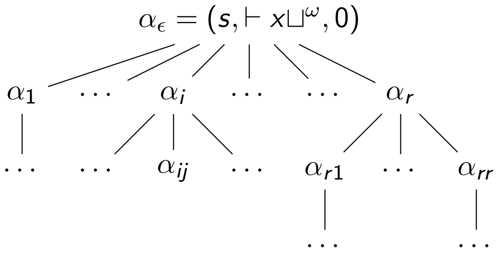

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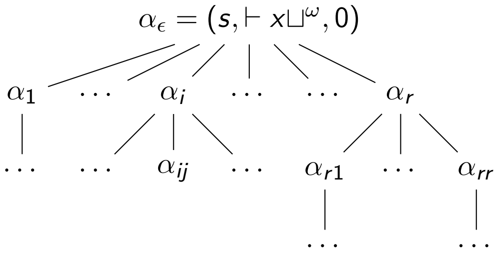

- If $r = \max_{q,X} |\Delta(q, X)|$ then the runs of M can be organized as an r-branching tree.
- $\alpha_{i_1 i_2 \cdots i_n}$ is the configuration of M after n-steps, where choice i_1 is taken in step 1, i_2 in step 2, and so on.
- Input x is accepted iff $\exists$ accepting configuration in tree.

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Observe that det(N) may not terminate if x is not accepted.

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- Tape 2, called simulation tape, will be used as N's tape when simulating N on a sequence of nondeterministic choices
- Tape 3, called choice tape, will store the current sequence of nondeterministic choices

Execution of det(N)

1. Initially: Input tape contains x, simulation tape and choice tape are blank
2. Copy contents of input tape onto simulation tape
3. Simulate N using simulation tape as its (only) tape
    1. At the next step of N, if state is q, simulation tape head reads X, and choice tape head reads i, then simulate the ith possibility in $\Delta(q, X)$; if i is not valid, then goto step 4
    2. After changing state, simulation tape contents, and head position on simulation tape, move choice tape's head to the right. If Tape 3 is now scanning $\sqcup$, then goto step 4
    3. If N accepts then accept and halt, else goto step 3(1) to simulate the next step of N.
4. Write the lexicographically next choice sequence on choice tape, erase everything on simulation tape and goto step 2.

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- If N accepts x then there is a sequence of choices that will lead to acceptance. det(N) will eventually have this sequence on choice tape, and then its simulation N will accept.
- If N does not accept x then no sequence of choices leads to acceptance. det(N) will therefore never halt!

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Robustness of the Class of TM Languages

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- Enhanced Turing Machine models: TM with 2-way infinite tape, multi-tape TM, nondeterministic TM, probabilistic Turing Machines, quantum Turing Machines ...
- Restricted Turing Machine models: queue machines, 2-stack machines, 2-counter machines, ...

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- To show that something can be solved on Turing machines, you can use any programming language of choice, *unless the problem specifically asks you to design a Turing Machine*

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- Directions L and R will be encoded as 0 and 1, respectively.

Turing Machine Codes

Continued

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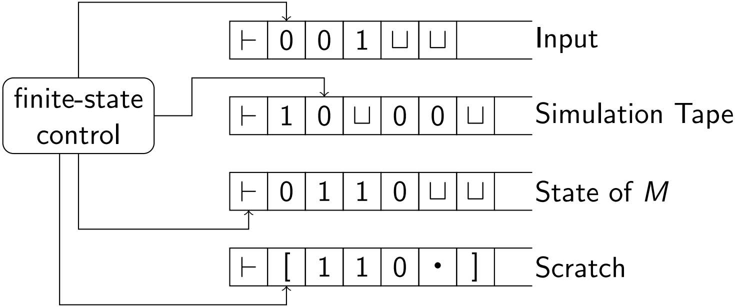
- We will denote the encoding of machine M and input x as $\langle M, x \rangle$

Featuring of the Encoding

- The precise choice of the alphabet and encoding is not important; it is merely to illustrate one precise encoding
  - In fact, when we write out TMs on paper using the english alphabet, punctuation marks, and set notation is perfectly good as well, as long as it is consistent.

Universal Turing Machine



Schematic picture of Universal TM

U will store the configuration of M by storing, the state of M on the state tape, and the tape of M on the simulation tape.

UTM algorithm

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5. If state on tape 3 is $0 \cdots 01$ then accept; if state is $0 \cdots 010$ then reject.