Tree Decompositions and Tree Width
\( \varepsilon \)-labeled ordered \( n \)-ary trees Structures

over the signature

\[ \mathcal{L} = (\lt, \exists s : \mathbb{N} \rightarrow, \exists \alpha : \mathbb{N} \rightarrow \varepsilon) \]

Doner, Thatcher-Wright Theorem. The set of \( \varepsilon \)-labeled trees definable in MSO is exactly the set of regular tree languages.

**Corollary** Consider a MSO sentence \( \varphi \).

Given any \( \varepsilon \)-labeled tree \( T \), the decision problem of determining if \( T \models \varphi \) is decidable in \( O(|T|) \).

**Proof** Construct a tree automaton \( A_{\varphi} \) corresponding to \( \varphi \).

On an input \( T \), run \( A_{\varphi} \) on \( T \) in linear time.

**MSO on Graphs.**

**Signature** \( \mathcal{L}_E = \varepsilon E^3 \)

**3-colorability** A graph \( G = (V, E) \) is 3-colorable if \( \exists c : V \rightarrow \{1, 2, 3\} \) s.t. 

\( u \neq v \in E, c(u) \neq c(v) \).
Theorem 3-colorability is NP-complete.

Define 3-colorability

\[ P_{3\text{col}} = \exists x_1 \exists x_2 \exists x_3 \]

"every vertex belongs to exactly one set"

\[ \land \forall x \forall y \forall z \land \exists \sum_{i=1}^{3} (\exists x_i \lor \exists x_i y) \]

Independent Set is \( I \subseteq V \) in \( G = (V, E) \) such that \( \forall u, v \in I, \exists u, v \notin E \).

Max Independent Set Given a graph \( G = (V, E) \) and \( k \in \mathbb{N} \), determine if there is an independent set of size \( \geq k \).

- NP-complete.

\[ P_{\text{ind}} = \exists I \exists x_1 \exists x_2 \ldots \exists x_k \]

\[ \land \land (x_i \neq x_j) \land \land I x_i \]

\[ \land \forall x \forall y, I x \land I y \rightarrow \exists x \exists y \]

Question Can these NP-complete problems be solved efficiently on more general graphs than just lines?

- Can we solve problems definable in...
For a graph $G = (V, E)$, a tree decomposition is a 2-$\ell$-labeled tree $T = (V_T, E_T, L_T)$

Node coverage $\forall u \in V, \exists t \in V_T, u \in L_T(t)$

Edge coverage $\forall \{u, v\} \in E, \exists t \in V_T, \{u, v\} \subseteq L_T(t)$

Coherence $\forall u \in V, \forall x, y \in V_T, x \neq y$

such that $u \in L_T(x) \cap L_T(y)$ then

$\forall z$ that appears on the unique path from $x$ to $y$ in $T$, $u \in L_T(z)$

- Every graph has a trivial tree-decomposition
- A graph may have many tree-decompositions

Definition: A tree decomposition $T = (V_T, E_T, L_T)$
of graph $G = (V, E)$ has width $w \in \mathbb{N}$ if

$$\forall t \in V, \quad |L_t(t)| \leq w + 1$$

Tree Width of graph $G = (V, E)$ is $w$ if there is a tree decomposition of $G$

of width $\leq w$.

**Proposition** Every tree $G = (V, E)$ has a tree decomposition of width 1.

**Proof**

$$T = (V_T, E_T, L_T)$$

$V_T = E$

$L_T(e = \langle u, v \rangle) = \langle u, v \rangle$

$(e_1, e_2) \subseteq E_T$ if $L_T(e_1) \cap L_T(e_2) \neq \emptyset$

**Proposition** If $G$ is a connected graph of width 1 then $G$ is a tree.

**Constructing Tree Decompositions of small width.**

**Problem** Given a graph $G = (V, E)$ and

$$1 \leq w \leq |V|$$

find $G$'s tree width $w$. 
Given a graph $G = (V,E)$, there is an algorithm $A$ that computes a tree decomposition $T$ of $G$ in time $T(f(W) \text{ poly } |V|)$, where $W$ is the tree width of $G$.

Let $T = (V_T, E_T, L_T)$ be a tree decomposition.

An edge $(t_1, t_2) \in E_T$ is redundant if $L_T(t_1) \subseteq L(t_2)$.

A tree decomposition is non-redundant if it has no redundant edges.

Proposition: Every graph $G$ of width $w$ has a non-redundant tree decomposition of width $w$. 

- Redundant edge $E_{t_1, t_2}$ can be contracted. [Remove vertices $t_1, t_2$ and add a vertex $E_{t_1, t_2}$]
a tree decomposition of width w that is binary.