April 25, 2025 12:24 PM

Approx Alg'ms

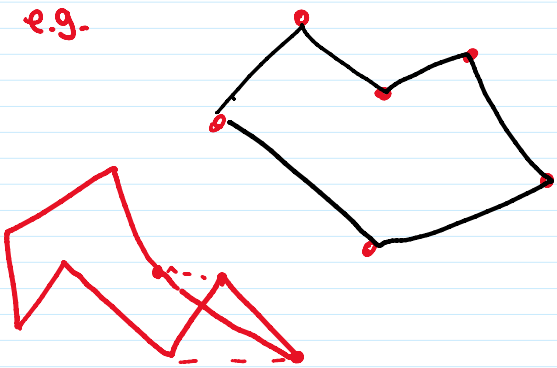
Problem (Traveling Salesman (TSP))

Given a weighted complete graph $d: V \times V \to \mathbb{R}^+$,

find a cycle C that visits every vertex [exactly once]

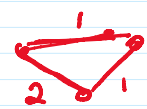
minimizing total weight $w(C) = \sum_{uv \in C} d(u,v)$.

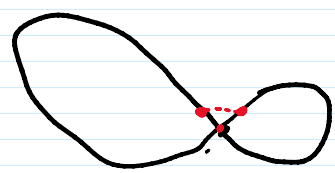
e.g.



e.g. V = points

d = Euclidean distance



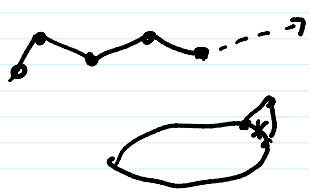


Assumption: d is a metric since

$d(u,v) = d(v,u)$

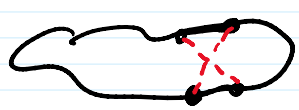
$d(u,v) \leq d(u,x) + d(x,v) \quad \forall x,u,v \in V$

heuristics: - greedy ("nearest neighbor")



$\Omega(\log n)$ approx factor in geom 2D.

- local improvement: "2-OPT"



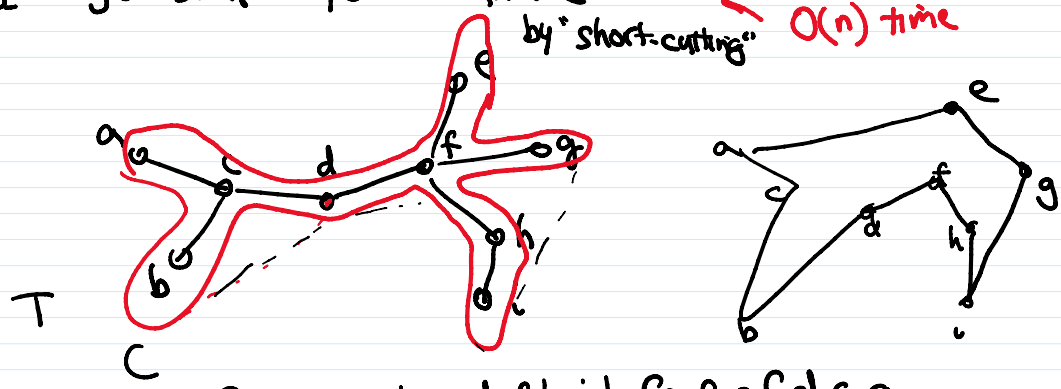
⋮

Min Spanning Tree

An MST-Based Alg'm

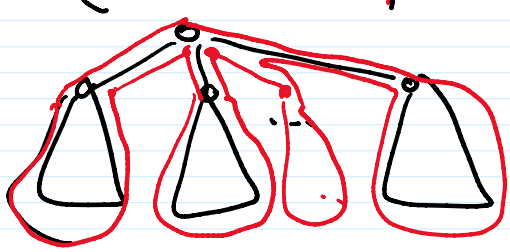
1. find MST T ← Prim ← $O(n \log n + m)$ time; Kruskal ← $O(m \log n)$ time; Borůvka
2. get closed walk C from T by traversing each edge twice
3. get simple cycle C′ from C by "short-cutting" ← $O(n)$ time

e.g.



C: a c b c d f h i h f g f e f d c a

C′: a c b d f h i g e a



Analysis of approx factor: $\boxed{2}$

let C^* be opt cycle.

By line 1, $w(T) \leq w(\underbrace{C^* - \{\text{an edge}\}}_{\text{path which is a tree}}) \leq w(C^*)$

By line 2, $w(C) = 2\,w(T)$

By line 3, $w(C') \leq w(C)$ by triangle ineq.

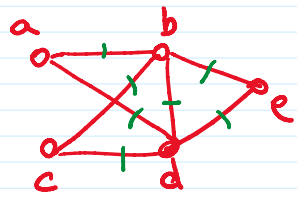
$\Rightarrow \quad w(C') \leq 2\,w(C^*)$. $\square$

Better?

Christofides' Alg'm (1976)

Def An <u>Euler tour</u> of a connected multi-graph is a closed walk that visits each edge exactly once.

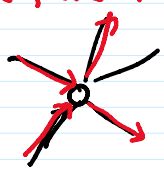
e.g.



abedbcda

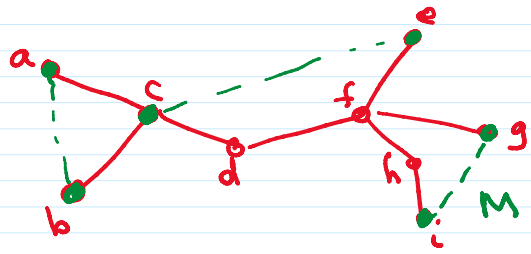
Fact $\exists$ Euler tour iff every vertex has even deg.

can be found in $O(m)$ time



1. find MST T
2. let $X = \{$vertex of odd deg in $T\}$
3. find min-weight perfect matching M of X ← polytime
   every vertex in X has deg 1 in M by defn of matching
4. find Euler tour C of $T \cup M$
   (by Fact)
   every vertex has even deg now!
5. get simple cycle C' from C.

e.g.



X

C: abcdfhigfeca

C': abcdfhigea

Correctness:

Obs $|X|$ is even.

Correctness:

Obs $|X|$ is even.
(# odd-deg vertices must be even).

Pf:

$$0 \equiv 2m = \sum_{v \in V} \deg(v) \equiv \cancel{(\#\text{even})} \cdot 0 + (\#\text{odd}) \cdot 1 \mod 2$$

$\square$

Obs $T \cup M$ is Eulerian.

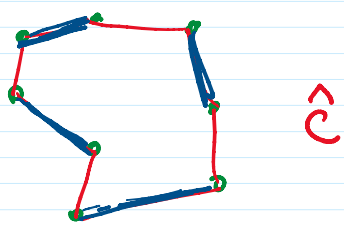
$$\forall v, \quad \deg_{T \cup M}(v) = \deg_T(v) + \deg_M(v)$$
$$= \begin{cases} \text{odd} + 1 = \text{even} & \text{if } v \in X \\ \text{even} & \text{if } v \notin X \end{cases}$$

$\square$

Analysis of approx factor:

By line 1, $w(T) \leq w(C^*)$.

Get $\hat{C}$ from C^* by removing vertices not in X.
Know $w(\hat{C}) \leq w(C^*)$ by triangle ineq.



Say $\hat{C} = v_1 v_2 v_3 \cdots v_{|X|} v_1$

Define 2 perfect matchings:

$$\hat{M}_1 = \{v_1 v_2, v_3 v_4, \ldots, v_{|X|-1} v_{|X|}\}$$
$$\hat{M}_2 = \{v_2 v_3, v_4 v_5, \ldots, v_{|X|} v_1\}$$

By line 2, $w(M) \leq w(\hat{M}_1)$
$w(M) \leq w(\hat{M}_2)$

$$\Rightarrow w(M) \leq \frac{w(\hat{M}_1) + w(\hat{M}_2)}{2} = \frac{w(\hat{C})}{2} \leq \frac{w(C^*)}{2}$$

By line 3, $w(C) = w(T) + w(M)$
$\leq \frac{3}{2} w(C^*)$

By line 4, $w(C') \leq w(C) \leq \frac{3}{2} w(C^*)$

$\Rightarrow$ approx factor $\leq \boxed{\frac{3}{2}}$

Arora'96 / Mitchell'96:

in Euclidean 2D case, $1+\varepsilon$.

"polytime approx scheme"

Karlin, Klein, Oveis Gharan'21:

in metric case, ~~1.49999...~~

$\frac{3}{2} - \frac{1}{10^{36}}$