Problem (Linear Programming (LP))

maximize/minimize a linear fn
s.t. linear inequality constraints over n real vars

i.e.
$$\begin{aligned} \max\ & c_1x_1 + \dots + c_nx_n \\ \text{s.t. } & a_{11}x_1 + \dots + a_{1n}x_n \le b_1 \\ & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n \le b_m \\ & x_1, \dots, x_n \ge 0 \end{aligned}$$
"standard form"

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \\ & x \ge 0 \end{aligned}$$

Ex farmer has 100 acres of land, loan \$60000, 600 units of storage

				profit
x_1	each acre of corn ⇒	cost \$300,	8 units of storage,	\$800
x_2	" " " soybean ⇒	cost \$800,	2 units,	\$900

$$\begin{aligned} \max\ & 800x_1 + 900x_2 \\ \text{s.t. } & x_1 + x_2 \le 100 \\ & 300x_1 + 800x_2 \le 60000 \\ & 8x_1 + 2x_2 \le 600 \\ & x_1, x_2 \ge 0. \end{aligned}$$

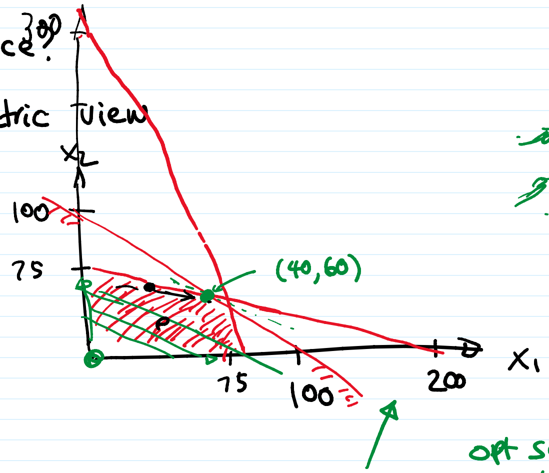
Appl'n / Special Case: max flow

($x_{u,v}$ = flow for (u,v))

$$\max \sum_{(s,v)\in E} x_{s,v}$$
$$\text{s.t. } \forall (u,v)\in E,\quad 0 \le x_{u,v} \le c(u,v)$$
$$\forall v \in V - \{s,t\},\quad \sum_{u:(u,v)\in E} x_{u,v} - \sum_{w:(v,w)\in E} x_{v,w} = 0$$

[note: each var appears in 2 constraints, one with coeff +1, one with coeff −1] brute force?

geometric view



$x_1 + x_2 \leq 100$

$8x_1 + 2x_2 \leq 600$

$3x_1 + 8x_2 \leq 600$

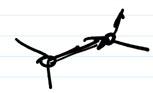
max $500x_1 + 600x_2$

$500x_1 + 600x_2 = c$

opt sol'n
$x_1 = 40$, $x_2 = 60$

equiv: find extreme vertex along dir. that is in intersection of m halfspaces in n dims

"feasible region"
a convex polyhedron in n dims



Simplex Algorithm (Dantzig '47)

idea - local improvement / hill climbing again!

```
v = initial vertex of feasible region
repeat {
    find an adjacent vertex v' of v
        in feasible region with larger objective value
    if none exists, return v
    v ← v'
}
```

Terms: a pt in feasible region ⟷ feasible sol'n
a vertex in feasible region ⟷ basic sol'n

n of the n+m constraints

finding an adj vertex ⟷ finding a pivot to swap in & out of the basis

Implementation Details

by example

Ex max $5x_1 + 4x_2 + 3x_3$

Ex

$$\max \ 5x_1 + 4x_2 + 3x_3$$
$$\text{s.t.} \quad 2x_1 + 3x_2 + x_3 \le 5$$
$$4x_1 + x_2 + 2x_3 \le 11$$
$$3x_1 + 4x_2 + 2x_3 \le 8$$
$$x_1, x_2, x_3 \ge 0$$

rewrite

$$\begin{aligned} \max \quad z &= 5x_1 + 4x_2 + 3x_3 \\ w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, \underbrace{w_1, w_2, w_3}_{\text{called slack vars}} \ge 0 \end{aligned}$$

initial sol'n: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (0,0,0,5,11,8)$

$\bar{z} = 0$.

Iteration 1: increase $\bar{x}_1$ to 2.5

x_1 enters basis, w_1 leaves

Substitute $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$

$$\begin{aligned} \max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.\ x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \ge 0 \end{aligned}$$

new sol'n: $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (2.5, 0, 0, 0, 1, 0.5)$

$\bar{z} = 12.5$

Iteration 2: increase $\bar{x}_3$ to 1

x_3 enters basis, w_3 leaves

$\vdots$