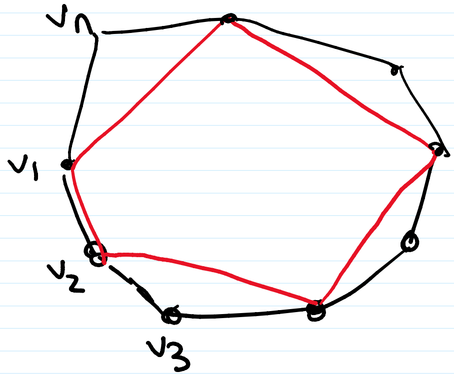
More DP

Max-Perimeter Subpolygon



$L = 5$

Given convex polygon $v_1, v_2, \dots, v_n, v_1$ and $L \le n$,
find subpolygon with L vertices
maximizing perimeter

DP Method 1

Define subproblems:

for each $1 \le i < j \le n$, $\ell \le L$,

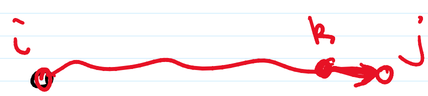
let $D^{(\ell)}(i,j)$ = max dist of path from v_i to v_j with ℓ links/hops in ccw order

Want
$$\max_{1 \le i < j \le n} \left(D^{(L-1)}(i,j) + d(v_j, v_i) \right).$$

$d(j,i)$

d ← Euclidean dist

Recursive formula:



$$D^{(\ell)}(i,j) = \max_{i < k < j} \left(D^{(\ell-1)}(i,k) + d(v_k, v_j) \right)$$

Base case: $D^{(1)}(i,j) = d(v_i, v_j)$

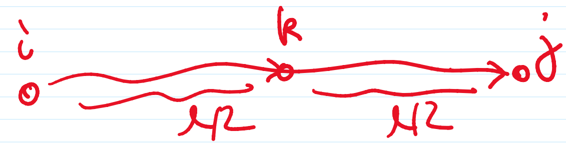
Analysis:

\# table entries $O(n^2 L)$

time per entry $O(n)$

$\Rightarrow$ total time $\boxed{O(n^3 L)} \leq O(n^4)$

DP Method 2



$$D^{(l)}(i,j) = \begin{cases} \boxed{\max\limits_{i<k<j} \left(D^{(l/2)}(i,k) + D^{(l/2)}(k,j) \right)} & \text{if } l \text{ even} \\ \max\limits_{i<k<j} \left(D^{(l-1)}(i,k) + \underbrace{d(v_k, v_j)}_{d(k,j)} \right) & \text{if } l \text{ odd} \end{cases}$$

"repeated squaring"

Only need $O(\log L)$ values of l

\# table entries $O(n^2 \log L)$

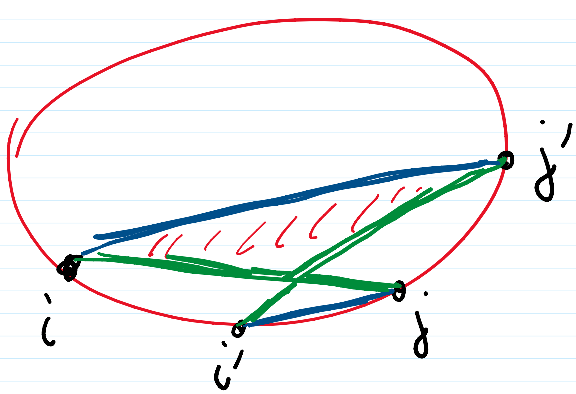
$\Rightarrow$ total time $\boxed{O(n^3 \log L)} \leq O(n^3 \log n)$

Improved DP Method 2

Def Function $d(\cdot,\cdot)$ satisfies convex Monge property

if $\forall\ i \leq i' \leq j \leq j'$,

$$d(i,j) + d(i',j') \geq d(i,j') + d(i',j).$$



Property is true for our problem by applying triangle ineq twice

Lemma Let $D(i,j) = \max_{i<k\leq j} \left(d(i,k) + d(k,j) \right)$

$K(i,j) =$ k that attains this max.

If d is convex Monge,

then K is monotonically increasing in each row & in each column.

(Also, D is convex Monge.)

DP Speedup Thm (F. Yao '82)

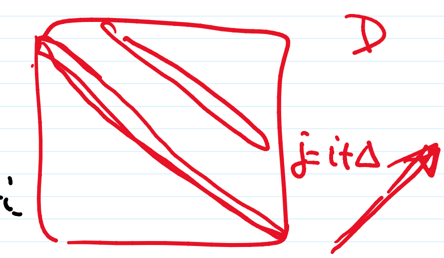
If d is convex Monge,

can compute D from d in $O(n^2)$ time.

Pf: Fix Δ.

Assume given $D(i, i+\Delta)$ $\forall i$.
$K(i, i+\Delta)$

Want to compute $D(i, i+\Delta+1)$ $\forall i$



Know $K(i, i+\Delta) \leq \underbrace{K(i, i+\Delta+1)} \leq \underline{K(i+1, i+\Delta+1)}$

$$D(i, i+\Delta+1) = \max_{K(i,i+\Delta) \leq k \leq K(i+1, i+\Delta+1)} \left(d(i,k) + d(k,j) \right)$$

total time $\displaystyle\sum_i \Big(\underbrace{K(i+1, i+\Delta+1)}_{f(i+1)} - \underbrace{K(i, i+\Delta)}_{f(i)} + 1 \Big)$

telescoping sum $\leq$ $O(n)$ per Δ

overall time $O(n^2)$. $\square$

$\Rightarrow$ $\boxed{O(n^2 \log L)}$ for max-perim subpolygon problem

time

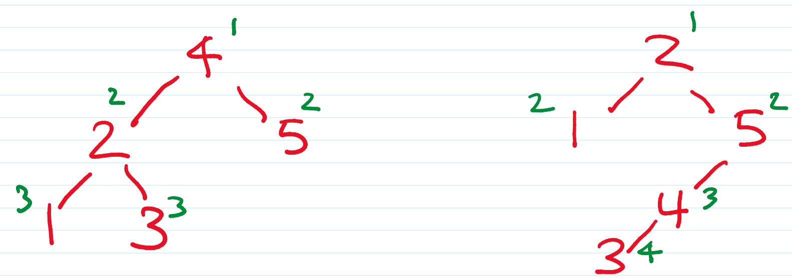
Rmk: Schieber '98 $O(n^{1+\varepsilon})$ time

Optimal Binary Search Tree

Given n elements $a_1, \dots, a_n$
their frequencies $f_1, \dots, f_n$,
build a binary search tree for $a_1, \dots, a_n$
that minimizes total search cost

i.e. $\sum_{i=1}^{n} f_i \cdot \mathrm{depth}(a_i)$.

e.g. a: 1, 2, 3, 4, 5
f: 4, 10, 1, 2, 8



cost $2\cdot 1 + (10+8)2 + (4+1)\cdot 3 = \cancel{55}\ 53$

cost $10\cdot 1 + (4+2)\cdot 2 + 2\cdot 3 + 1\cdot 4 = \cancel{44}.\ 32$