Matrix Multiplication

Given $n \times n$ matrices A, B,

compute $C = AB$.

e.g.
$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}}_{B} = \begin{pmatrix} 36 & 42 & x \\ x & x & x \\ x & x & x \end{pmatrix}$$

obvious alg'm: (by def'n)

for $i := 1$ to n

 for $j := 1$ to n

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$\Rightarrow \boxed{O(n^3)}$ time better?

(lots of appl'ns: most basic nontriv. matrix op

inverse/det/ $Ax = b$ . . .)

(addition: $O(n^2)$ time)

Strassen's Alg'm ('69)

divide $A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ $B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$

where $A_{11}, A_{12}, A_{21}, A_{22}$ are $\frac{n}{2} \times \frac{n}{2}$ matrices
$B_{11}, B_{12}, B_{21}, B_{22}$

$$AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right) = \left(\begin{array}{c|c} C_1 + C_2 & C_5 + C_6 + C_3 - C_2 \\ \hline C_5 + C_7 + C_1 - C_4 & C_3 + C_4 \end{array}\right)$$

(note: $AB \neq BA$)

$$\Rightarrow T(n) = 8\,T\left(\frac{n}{2}\right) + O(n^2)$$

(4 matrix additions)

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$$\Rightarrow \quad O(n^{\log_2 8}) = \boxed{O(n^3)}$$

not faster

more clever idea.

$$C_1 = A_{11}(B_{11} - B_{21}) \qquad C_2 = (A_{11} + A_{12}) B_{21}$$

$$C_3 = A_{22}(B_{22} - B_{12}) \qquad C_4 = (A_{21} + A_{22}) B_{12}$$

$$C_5 = (A_{11} + A_{22})(B_{21} + B_{12})$$

$$C_6 = (A_{12} - A_{22})(B_{21} + B_{22})$$

$$C_7 = (A_{21} - A_{11})(B_{11} + B_{12})$$

$$\Rightarrow \quad T(n) = 7\,T\left(\frac{n}{2}\right) + O(n^2)$$

$$\Rightarrow \quad O(n^{\log_2 7}) \leq \boxed{O(n^{2.81})}$$

Better?

3-way D&C: $T(n) = \underbrace{23}\,T\left(\frac{n}{3}\right) + O(n^2)$

$$\Rightarrow \quad O(n^{\log_3 23}) \leq O(n^{2.85})$$

⋮

Pan '78: $T(n) = \underbrace{143640}_{}\,T\left(\frac{n}{70}\right) + O(n^2)$

$\frac{k^3-4k}{3} + 6k^2$; $\frac{n}{k}$

$$\Rightarrow \quad O(n^{\log_{70} 143640}) \leq O(n^{2.796})$$

Pan '79: $O(n^{2.781})$

Bini et al. '80: $O(n^{2.780})$

Schönhage '81: $O(n^{2.522})$

Strassen '86: $O(n^{2.479})$

Coppersmith, Winograd '90: $O(n^{2.376})$

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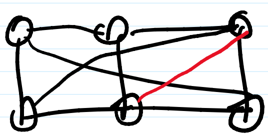
Strothers '10: } $O(n^{2.374})$
Vassilevska W. '12: } $O(n^{2.373})$

⋮

Duan, Wu, Zhou '23: $O(n^{2.3719})$
Vassilevska, Xu, Xu, Zhou '24: $O(n^{2.3716})$

matrix mult. exponent $\underline{\omega}$

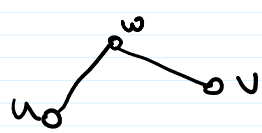
Appl'n: Given undir. graph $G=(V,E)$ with n vertices, decide $\exists$ cycle of $\underbrace{\text{length } 3}_{\text{called triangle}}$



~~no~~ yes

trivial alg'm: $O(n^3)$

let $a_{uv} = \begin{cases} 1 & \text{if } uv \in E \\ 0 & \text{else} \end{cases}$ (adj matrix)



for $u \in V$ do
  for $v \in V$ do
    compute $c_{uv} = \sum_{w \in V} a_{uw} a_{wv}$
    if $c_{uv} > 0$ & $uv \in E$ return yes
return no

$$\Rightarrow O(n^{2.3716} + n^2) = \boxed{O(n^{2.3716})}$$

next time:

Dynamic programming