cs473 Algorithms

Problem Set #9

Prof. Michael A. Forbes — Due: *Fri., 2024-04-19 17:00*

All problems are of equal value.

1. Consider the polyhedra $P \subseteq \mathbb{R}^3$ defined by the equations

$$\begin{aligned} x + 2y - z &\leq 0 \\ 3x + y + z &\leq 11 \\ -x + 2y &\leq 5 \\ x, y, z &\geq 0 \end{aligned}$$

   (a) Using Fourier-Motzkin elimination, give equations for the projection of P onto the (x, y)-plane; that is, eliminate z.

   (b) Using Fourier-Motzkin elimination, give equations for the projection of P onto the (y, z)-plane; that is, eliminate x.

2. Monotone Satisfiability. Kleinberg-Tardos Chapter 8, Problem #6.

3. This problem will relate pairwise independence and linear programming.

   (a) Consider a bounded polyhedra with m non-axis constraints, n axis constraints, and n variables. That is, $P = \{x : Ax \leq b, x \geq 0\} \subseteq \mathbb{R}^n$ for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, where the constraints on P also imply that for some $B > 0$ that $x \leq B$ for all feasible x.

   Suppose that $m < n$. Prove that P contains a point where only $n - m$ coordinates are non-zero.

   Hint: what can a vertex of P look like?

   (b) Consider an arbitrary probability distribution over n binary variables $\mathsf{X}_1, \ldots, \mathsf{X}_n \in \{0, 1\}$. Equivalently, we have a probability distribution over $\{0, 1\}^n$. One can interpret this distribution as a list of 2^n probabilities. Write a system of linear constraints that defines this set of probabilities as a polyhedron.

   (c) Now further consider such probability distributions that are pairwise independent. That is, for all $1 \leq i < j \leq n$ and $b, b' \in \{0, 1\}$, $\Pr[\mathsf{X}_i = b \wedge \mathsf{X}_j = b'] = 1/4$. Write a set of linear inequalities that defines this set of probability distributions as a polyhedron.

   (d) Using the above parts, conclude that there exist pairwise independent probability distributions that are distributions over $\leq O(n^2)$ n-bit vectors.

   To compare, an general probability distribution will involve all 2^n n-bit vectors.

   Hint: minimize the number of non-axis constraints by removing redundant constraints.