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CS473 Algorithms: Lecture 9 (2024-02-13)

logistics: - pset 3 due F17
- exam 1 02-26

last lecture: flows
- review def
- cuts
- max flow ≤ mincut
- ——"—— = ——"—— iff G^f no $s \to t$ path
- ——"—— iff FF correct
- ——"—— for $\boxed{\text{integral}}$ capacities

today: flows

[previously] reading: KT 7.3

[recap] Q: compute max flow?

idea: repeatedly augment flow, via augmenting paths in residual graphs

alg: (Ford-Fulkerson):
- $f_e \leftarrow 0$, $e \in E$ [init w/ zero flow]
- init G^f [residual graph]
- while exists augmenting path p in G^f
  - $f \leftarrow f + p$
  - $G^f \leftarrow G^{f+p}$
- return f

prop: $\boxed{\text{any}}$ flow f, $|f| \le \sum_e c_e = F$

prop: Ford Fulkerson takes $O(F)$ iterations, $O(mF)$ time [increase flow >0 each iteration] [integrality ⇒ ≥ 1] [[limit by]] [time per-iteration]

def: (s,t) cut C is partition $V = S \sqcup T$, $S \ni s$, $T \ni t$
capacity $|C| = \sum_{\substack{e: u \to v \\ u \in S,\, v \in T}} c_e$

def: f flow, $S \subseteq V$, flow through S is $f(S) = \underbrace{f^{out}(S)}_{= \sum_{\substack{e: u \to v \\ u \in S,\, v \notin S}} f_e} - \underbrace{f^{in}(S)}_{= \sum_{\substack{e: u \leftarrow v \\ u \in S,\, v \notin S}} f_e}$

prop: f (s,t) flow, $C = (S,T)$ (s,t)-cut, then
$|f| = f(s) = \sum_{v \in S} f(v) = f(S) = \underbrace{f^{out}(S)}_{\le |C|} - \underbrace{f^{in}(S)}_{\ge 0} \le |C|$ [source flow] [conservation] [? ?] [$f_e \le c_e$]

$\Rightarrow \max_f |f| \le \min_C |C|$

Q: ——"—— = ——"—— ? [mathematically interesting] [algo applications]

prop: f flow in G. $S = \{u : s \leadsto u \text{ in } G^f\}$. equiv:
(1) G^f has no $s \leadsto t$ path
(2) $C = (S, V \setminus S)$ is (s,t)-cut w/ $|C| = |f|$
(3) f is max flow

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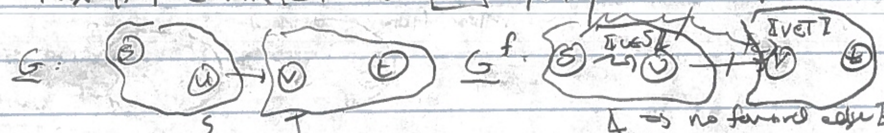
sketch: $\neg(1) \Rightarrow \neg(3)$: $|f+p| \geq |f|$

(2) $\Rightarrow$ (3): max $|f| \leq$ min $|C|$ $\Rightarrow$ [equality then both opt]

(1) $\Rightarrow$ (2):



[$\Rightarrow$ no forward edges]
$\Rightarrow f_e = c_e$

$\Rightarrow f_e = 0$

$\therefore$ [saturate $S \to T$ flow, no $T \to S$ flow]
$\Rightarrow |f| = |C|$
[no $s \leadsto t$ path in G^f]

· vs 2022

cor: FF terminates w/ max flow

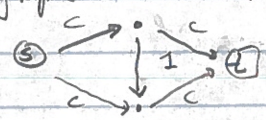
thm [max flow min cut]: G w/ $\boxed{\text{integral}}$ capacities, $\max_{f\ (s,t)\text{-flow}} |f| = \min_{C\ (s,t)\text{-cut}} |C|$

[and pseudo] $\Leftarrow$ Q: are we done?

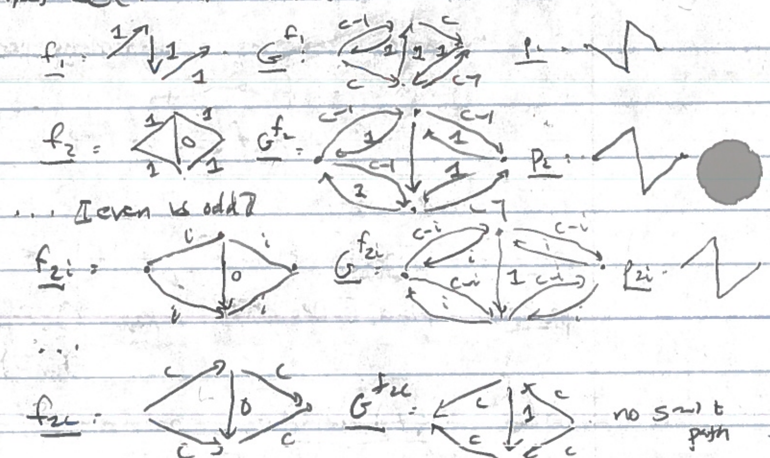
· give P_θ

clm: exists graphs where FF takes $\Omega(F)$ iterations

pf:

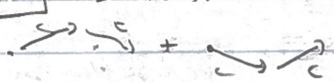


$F = \Theta(c)$



$2c$ iterations

rmk: $\boxed{\text{but}}$ 2 iterations suffice



$\Rightarrow$ some augmenting path choices are $\boxed{\text{bad}}$

def: for problem on n integers $a_1, \ldots, a_n$, algo is polynomial time if it runs in poly($\Sigma \lg a_i$) time [a_i in binary]
[unspecified poly]

pseudo ___ poly(Σa_i) time [a_i in unary]
[unary 128] [binary 8]

rmk: - often pseudopoly not efficient: 10000000 vs 10000000

- sometimes pseudo poly can be reasonable
  - knapsack DP was pseudo poly

cor: max flow [and min cut] has pseudopoly time algo

$\Leftarrow$ Q: polytime?
[FF good if start close]

· notation f^* vs OPT

prop: given f flow in G, w/ $|f| \geq |f^*| - B$, can find max flow [max flow] in $O(m \cdot B)$ time

sketch: run FF starting w/ f, only $\leq B$ iterations needed $\leftarrow$ integrality

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cor: max flow in $O(m \cdot |f^*|)$ time

rmk: many natural problems have $|f^*| \leq \mathrm{poly}(n,m)$ [next lecture] [$\Rightarrow$ polytime]

Q: polytime algo in $\boxed{\text{general}}$?

idea: find $\boxed{\text{good}}$ augmenting paths [FF allows $\boxed{\text{any}}$ choice] [many choices possible] [saw example of $\boxed{\text{bad}}$ choices]

$\hookrightarrow$ large value

idea: reduce to

def: f flow in G, $\Delta \in \mathbb{N}$ [param] [modify G^f] the Δ-bottleneck residual graph $G^{f,\Delta}$ is

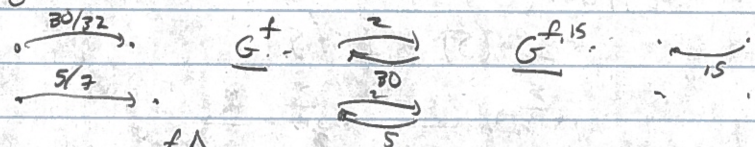
$$V^{f,\Delta} = V$$

$$E^{f,\Delta} = \{ e \in E^f,\ (c^f_e) \geq \Delta \}$$ [discard $\boxed{\text{small}}$ edges]

$$(c^{f,\Delta})_e = (c^f)_e \quad \text{for } e \in E^{f,\Delta}$$

lem: $G^{f,1} = G^f$ [integral capacities]

eg: G = , G^f = , $G^{f,15}$ =



prop: $s \rightsquigarrow t$ paths in $G^{f,\Delta}$

- are $s \rightsquigarrow t$ paths in G^f [$E^{f,\Delta} \subseteq E^f$] [so can use for FF]
- are augmenting paths p w/ $|p| \geq \Delta$ [all edges residual capacity $\geq \Delta$]

Q: what is Δ?

idea [scaling]: start with large Δ, make smaller over time

algo (scaling FF):

- init f, G^f [as before]
- $F = \sum_e c_e$ [bound on max flow]
- $\Delta = 2^{\lfloor \lg F \rfloor}$
- init $G^{f,\Delta}$
- while $\Delta \geq 1$
  - while $s \rightsquigarrow t$ path p in $G^{f,\Delta}$
    - $f \leftarrow f + p$
    - update $G^f, G^{f,\Delta}$
  - $\Delta \leftarrow \Delta/2$
- return f

prop: scaling FF terminates w/ maxflow

sketch: scaling FF instantiates FF w/ rule for choosing "good" augmenting path

eventually $\Delta = 1 \Rightarrow$ all augmenting paths considered in $G^{f,1} = G^f$ vs 2022

$\Rightarrow$ any run of scaling-FF [is] run of FF $\leftarrow$ terminates w/ maxflow

Q: complexity?

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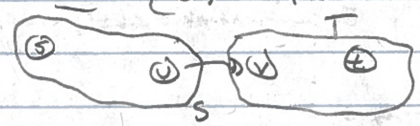
idea: use max-flow = min-cut

prop: f flow in G. $G^{f,\Delta}$ no $s \rightsquigarrow t$ path $\Rightarrow |f| \geq |f^*| - m(\Delta - 1)$
[$\Rightarrow$ "close" to opt]
[$\Rightarrow$ "few" iterations off to reach opt]
["=" if $\Delta = 1$]

sketch: [repeat max-flow = min cut proof]
$S = \{v : s \rightsquigarrow v \text{ in } G^{f,\Delta}\}$

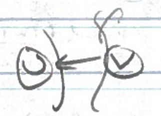
clm: $(S, T = V \setminus S)$ is (s,t)-cut [$s \in S$, $t \in T$] [as before]

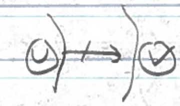
G: 

$G^{f,\Delta}$: 

[$\Rightarrow$ no forward edge]

$\Rightarrow \underset{c_e - f_e}{(c^f)_e} < \Delta \Rightarrow f_e \geq c_e - (\Delta - 1)$

G. 

$G^{f,\Delta}$. 

[$\Rightarrow$ no backward edge]

$\Rightarrow \underset{= f_e}{(c^f)_e} < \Delta \Rightarrow f_e \leq \Delta - 1$

$$\Rightarrow |f| = f(s) = f(S)$$
$$= f^{out}(S) - f^{in}(S)$$
$$= \sum_{\substack{e: u \to v \\ u \in S,\, v \in T}} \underbrace{f_e}_{\geq c_e - (\Delta - 1)} - \sum_{\substack{e: u \leftarrow v \\ u \in S,\, v \in T}} \underbrace{f_e}_{\leq \Delta - 1}$$
$$\geq \underbrace{\sum_{\substack{e: u \to v \\ u \in S,\, v \in T}} c_e}_{= |c(S,T)| \geq |f^*|} - \underbrace{\sum_e (\Delta - 1)}_{= m(\Delta - 1)} \quad \text{[all } m \text{ edges]}$$

∎

cor: f flow in G. no $s \rightsquigarrow t$ path in $G^{f,2\Delta}$ $\Rightarrow$ any sequence of $(\geq \Delta)$-value augmentations to f has $\leq 2m$ paths

pf: no $s \rightsquigarrow t$ path in $G^{f,2\Delta} \Rightarrow |f| \geq |f^*| - 2m\Delta$

each $(\geq \Delta)$ value augmentation adds $\geq \Delta$ flow

$\Rightarrow$ k augmentations add $\geq k\Delta$ flow and $|f| + k\Delta \leq |f^*| \leq |f| + 2m\Delta$ $\Rightarrow k \leq 2m$ [polynomial!] ∎

cor: scaling FF runs in $O(m^2 \lg F)$ time

pf: algo: [recall] $\Delta = 2^{\lfloor \lg F \rfloor}$
while $\Delta \geq 1$
while $s \rightsquigarrow t$ path in $G^{f,\Delta}$
augment
$\Delta \leftarrow \Delta/2$

$O(\lg F)$ iterations
$\sum$?
$O(m)$ rect [seen]

return f

clm: $O(m)$ augmentations per round

pf: $\Delta = 2^{\lfloor \lg F \rfloor}$ [first round]
[flow] $2\Delta > F \geq c_e$, [any e] no edges in $G^{f,2\Delta}$ [Δ too large] $\Rightarrow$ no $s \rightsquigarrow t$ path in $G^{f,2\Delta}$ [above] $\Rightarrow \leq 2m$ augmentations in $G^{f,\Delta}$

$\Delta = 2^i$. [later rounds] end of last round: no $s \rightsquigarrow t$ path in $G^{f,2\Delta}$ [Δ was halved]
[above] $\Rightarrow \leq 2m$ further augmentations [this round] in $G^{f,\Delta}$ [$\geq \Delta$ in value] ∎

vs2022

vs2022

today: flows
- review
- FF pseudopoly
- $\boxed{\text{bad}}$ augmenting paths
- scaling FF polytime [large value paths]

reading: KT 7-3

next lecture: flows

logistics: pset 3 due F17