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cs473 Algorithms : lecture 11 (2024-02-20)

logistics : - pset 4 due F17
- exam 1 : 2024-02-26 19:00
- piazza - details
- practice
- thursday exam review

revise .

last lecture : - flows - bipartite matching [[def]]
- [reducing] it to max flow [[don't start from scratch]]

revise .

today : flows

reading : KT 7.7, 7.8

[[last lecture]] [[phrase differently]]

thm : bipartite perfect matching in $O(nm)$ time

[[bipartite]]
[[perfect matching]]
[[other edges]]

[[reduction]]

idea $G=(L \cup R, E)$ $\longmapsto$ $G'=(V', E')$

Q : perfect matching in G? [[∃ part]] only if $|L|=|R|$

Q : max flow in G' value $|L|$? $|R|$?

sketch :

[[max flow]]

"forward" reduction : creating [new] problem

vs pset 4 .

G n vertices, m edges $\to$ G' n' vertices, m' edges in time $T(n,m)$

$O(n)$ $O(n+m)$ $O(n+m)$ [[forward]] [[need efficiency]]

G has perfect matching $\Rightarrow$ G' has flow value $|L|$

[[solve Max flow]]

solve new problem :

$T'(n', m')$ to solve new problem

$= O(n'm') = O(nm)$ [[T' big ⇒ slow]] [[n', m' big → slow]]

reexpress in [original] parameters

[[recover matching from flow]]

"backwards" reduction : solving [original] problem via soln to new problem [[backwards]]

G has perfect matching $\Leftarrow$ G' has [integral] flow value $|L|$

$T''(n, m, n', m') = O(m)$

correctness : clear

complexity : creating new instance $O(n+m)$ [[translation forward]] [[↳ computationally simple]]

solving ——— " ——— $O(nm)$ [[Main part]]

recovering soln to orig instance $O(m)$ [[translation back]]

[[bipartite matching is [special] case]]

Q : solve more [general] flow problems?

def : capacitated graph w/ [demands] is - capacitated graph $G=(V,E)$

[[edge]] capacities $(c_e)_{e \in E}$ [[>0]]

[[vertex]] demands $d=(d_v)_{v \in V}$ over [$\mathbb{Z}$] [[positive or neg]]

a circulation is a flow $f=(f_e)_e$ s.t.

capacity constraints : $0 \le f_e \le c_e$ $\forall_e$

conservation constraints : $f^{in}(v) - f^{out}(v) = d_v$ $\forall_v$

the circulation problem is to decide if feasible circulation [exists]. [[is "base"]]

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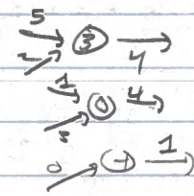
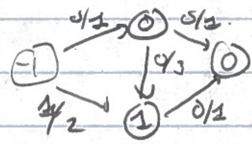
rmk? **rmk:** conservation constraints

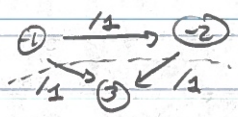
$$f^{in}(v) - f^{out}(v) = d_v \quad \begin{array}{ll} >0 & \text{is } \underline{\text{demand}} \\ =0 & \text{is } \underline{\text{conservation}} \\ <0 & \text{is } \underline{\text{supply}} \end{array}$$



figs **eg:**



is feasible



is infeasible
$\Uparrow$
cut value 2 < demand

lem: feasible circulation exists $\Rightarrow$ $\sum_v d_v = 0$ [[total conservation]]

pf:
$$\begin{aligned} 0 &= f^{out}(V) - f^{in}(V) \quad \text{[[no edges]]} \\ &= f(V) \quad \text{[[def of net flow]]} \\ &= \sum_{v \in V} [\underbrace{f^{out}(v) - f^{in}(v)}_{=-d_v}] \quad \text{[[as before, double counting edges]]} \end{aligned}$$
[[conservation constraints]]

thm: circulation feasibility in $O(nm)$ time [[max flow time]]

If all demands, capacities are integral, circulation feasible $\Rightarrow$ $\boxed{\text{integral}}$ circulation exists

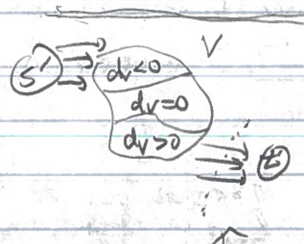
expand **pf: idea:** reduce to max flow [[solve general problem reduces to less general problem]]

G capacitated graph w/ demands d

construction: capacitated graph $G' = (V', E')$ [[forward reduction]]

$$V' = \{s'\} \text{ [[no source in circulation]]} \cup V \cup \{t'\}$$

$$E' = E$$
$$\cup \{(s', v) : d_v < 0\} \quad \text{[[provide supply]]}$$
$$\cup \{(v, t') : d_v < 0\} \quad \text{[[consume demand]]}$$



$$c'_{e'} = \begin{cases} c_{e'} & e' \in E \\ -d_v \boxed{>0} & e' = (s', v) \quad v \in V \quad d_v < 0 \\ d_v \boxed{>0} & e' = (v, t') \quad v \in V \quad d_v > 0 \end{cases}$$
[[clear]]

clm: G' has $O(n)$ vertices, $O(n+m)$ edges, constructible in $O(n+m)$ time [[forward]]

G has feasible circulation $\to$ G' has max flow value $\sum_{v: d_v > 0} d_v = D$

pf: clm: max flow $G' \leq D$
pf: via cut

clm: max flow $G' \geq D$
pf: circulation f in G, create f' in G', $f'_{e'} = \begin{cases} f_{e'} & e' = (u,v) \in E \\ -d_v & e' = (s', v) \quad d_v < 0 \\ d_v & e' = (v, t') \quad d_v > 0 \end{cases}$
[[fig]]

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clm: f' valid flow

pf: capacity: $\underset{E}{e \in}\ u \xrightarrow{f_e/c_e} v \mapsto e' \in E'$ ——"—— [edge constraint unchanged]

$s' \xrightarrow{-d_v/d_v} v$ [new edges]

$v \xrightarrow{d_v/d_v} t'$

[f circulation]

[no constraint on s', t']

conservation: $v \in V$ $\quad f^{in}(v) - f^{out}(v) = d_v$

$d_v < 0$: $\quad \Rightarrow (f')^{in} = f^{in} - d_v = f^{out}$

$(f')^{out} = f^{out}$

$\Rightarrow f'(v) = 0$ [net flow]

$d_v > 0$: ——"——

clm: f' has value $D = \sum_{v: d_v > 0} d_v$

pf: $s' \xrightarrow{-d_v} v_1$ $\quad f'(s') = \sum_{v: d_v < 0} -d_v = \sum_{v: d_v > 0} d_v = D$

$\sum_v d_v = 0$ ∎

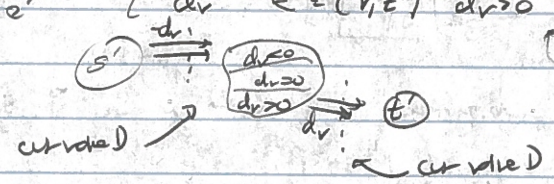
[backwards]

clm: G has $\boxed{\text{integral}}$ feasible circulation $\Leftarrow$ G' has $\boxed{\text{integral}}$ max flow value $D = \sum_{v: d_v > 0} d_v$ [max possible value]

pf: integral flow f' in G' [$\hookrightarrow$ $\boxed{\text{arbitrary}}$ such flow]

clm: $f'_{e'} = \begin{cases} -d_v & e' = (s', v) \quad d_v < 0 \\ d_v & e' = (v, t') \quad d_v > 0 \end{cases}$ [saturated edges]

sketch:



$|f'| = D \Rightarrow$ cuts are saturated $\Rightarrow$ edges saturated $\Rightarrow$ ∎

create circulation f in G by: $f_e = f'_e$ $\quad e = (u,v) \in E$ [drop new edges]

clm: f integral [clear], feasible

pf: capacity: clear

conservation: $d_v < 0$: [as above]

$(f')^{in}$ $\quad (f')^{out} = f^{out}$ $\quad$ recheck

$(f)^{in}$ $\quad -d_v$ $\quad s'$

$f^{in} = \underbrace{(f')^{in}}_{} + d_v$

$= (f')^{out} = f^{out}$

$\Rightarrow f^{in} - f^{out} = d_v$

$d > 0$: ——"—— ∎

algo: construct G' from G | $O(n)$

solve $\boxed{\text{integral}}$ max flow in G' | $O(n'm') = O(nm)$ [new vs old params]

reconstruct $\boxed{\text{integral}}$ circulation in G | $O(m)$

check flow value $= D$

correctness: clear [assuming]

complexity: ∎

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= [more?]

def: capacitated graph w/ $\boxed{\text{demands}}$ and $\boxed{\text{lower bounds}}$. capacitated graph $G=(V,E)$

capacities $(c_e)_{e\in E}$, $c_e>0$
lower-bounds $(l_e)_{e\in E}$ $0\le l_e\le c_e$
demands $d_{v\in E}$

a circulation is a flow $(f_e)_e$ w/
capacity constraints $0\le l_e\le f_e\le c_e$ $\forall e\in E$
—— " —— [as before]
conservation:
[as before]

circulation problem is —— " ——

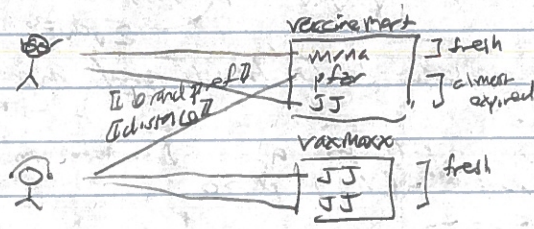
thm: circulation feasibility $\boxed{\text{w/ lower bounds}}$ solvable in $O(nm)$ time
[as before]
↳ w/ integrality —— " ——

sketch: reduce to circulations w/ $\boxed{\text{no}}$ lower bounds

clm: f feasible in G iff $f=f^\ell+\tilde{f}$, f^ℓ has $(f^\ell)_e=l_e$ [fixed]
$\tilde{f}$ circulation in $\tilde{G}$ ← $\boxed{\text{no}}$ lower bounds
$\tilde{G}$ has: capacities $0\le\tilde{f}_e\le c_e-l_e$ [excess]; $l_e=(f^\ell)_e$
conservation: $\tilde{d}_v=d_v-f^\ell(v)$ [net flow]

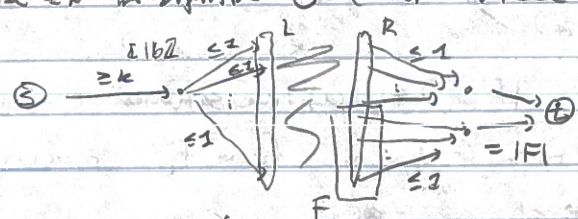
[I do reduction: -forward -solve -backward]

Q: can we use all nearly expired vaccine doses?



Q: matching size $\ge k$ in bipartite $G=(L\cup R,E)$ where $F\subseteq R$ are matched?

[by picture]



[analysis is clear]

= reductions to maxflow [I forward, solve maxflow, backward I]   [bipartite matching]

today: flows
- circulations w/ demands [reduce to maxflow]
- —— " —— w/ lower bounds [reduce to no lower bounds]
- bipartite matching w/ forced matches [lbs in maxflow]

reading: KT 7.7, 7.8

next lecture: - exam review

logistics: - pset 4 due F17 [graded?]
- exam1 02-26 19:00