More NP-Complete Problems

Lecture 22

April 27, 2021

Most slides are courtesy Prof. Chekuri

Recap

NP: languages/problems that have polynomial time certifiers/verifiers

A problem X is **NP-Complete** iff

- X is in **NP**
- X is **NP-Hard**.

X is **NP-Hard** if for every Y in **NP**, $Y \leq_P X$.

Theorem (Cook-Levin)

SAT *is* **NP-Complete**.

Recap contd

Theorem (Cook-Levin)

SAT *is* **NP-Complete**.

Established **NP-Complete**ness via reductions:

1. **SAT** is **NP-Complete**.
2. **SAT** $\leq_P$ **3-SAT** and hence 3-SAT is **NP-Complete**.
3. **3-SAT** $\leq_P$ **Independent Set** (which is in **NP**) and hence **Independent Set** is **NP-Complete**.
4. **Clique** is **NP-Complete**
5. **Vertex Cover** is **NP-Complete**
6. **Set Cover** is **NP-Complete**
7. **Subset Sum** is **NP-Complete**

Today

Prove

- **Hamiltonian Cycle** is **NP-Complete**
- **3-Coloring** is **NP-Complete** (self-study)

All via reductions from **3-SAT**

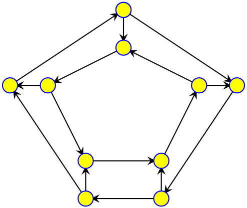
Part I

NP-Completeness of Hamiltonian Cycle

Directed Hamiltonian Cycle

Input Given a directed graph $G = (V, E)$ with n vertices

Goal Does G have a Hamiltonian cycle?



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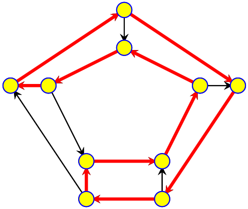
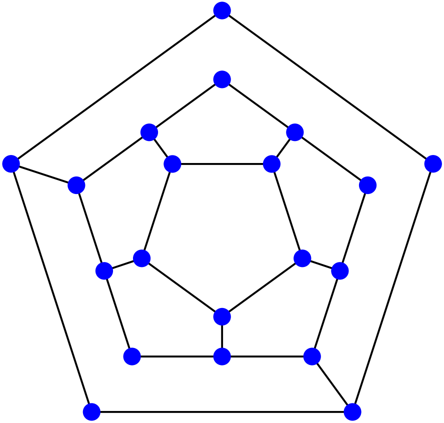


Is the following graph Hamiltonianan?



(A) Yes.

(B) No.

Directed Hamiltonian Cycle is NP-Complete

- Directed Hamiltonian Cycle is in ***NP***
  - Certificate: Sequence of vertices
  - Certifier: Check if every vertex (except the first) appears exactly once, and that consecutive vertices are connected by a directed edge
- Hardness: We will show
  3-SAT $\leq_P$ Directed Hamiltonian Cycle

Reduction

Given 3-SAT formula φ create a graph G_φ such that

- G_φ has a Hamiltonian cycle if and only if φ is satisfiable
- G_φ should be constructible from φ by a polynomial time algorithm $\mathcal{A}$

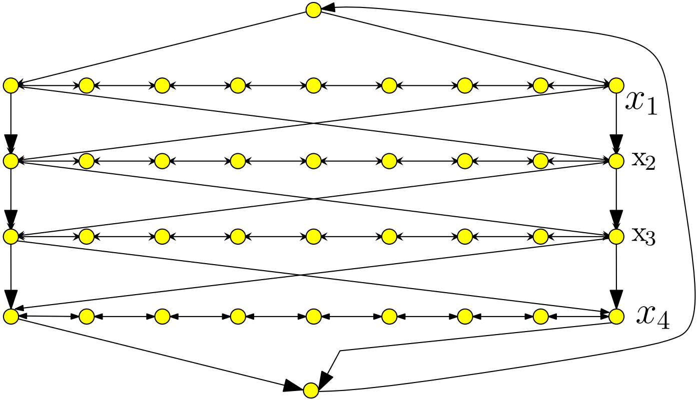
Notation: φ has n variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$.

Reduction: First Ideas

- Viewing SAT: Assign values to n variables, and each clauses has 3 ways in which it can be satisfied.
- Construct graph with 2^n Hamiltonian cycles, where each cycle corresponds to some boolean assignment.
- Then add more graph structure to encode constraints on assignments imposed by the clauses.

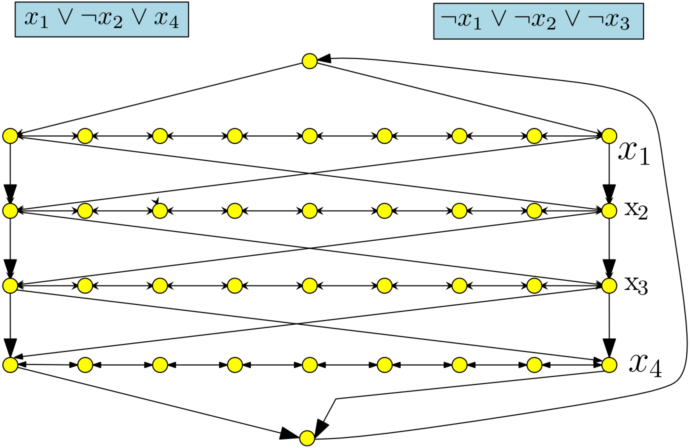
The Reduction: Phase I

- Traverse path i from left to right iff x_i is set to true
- Each path has $3(m+1)$ nodes where m is number of clauses in φ; nodes numbered from left to right (1 to $3m+3$)



The Reduction: Phase II

- Add vertex c_j for clause C_j. c_j has edge *from* vertex $3j$ and *to* vertex $3j+1$ on path i if x_i appears in clause C_j, and has edge *from* vertex $3j+1$ and *to* vertex $3j$ if $\neg x_i$ appears in C_j.



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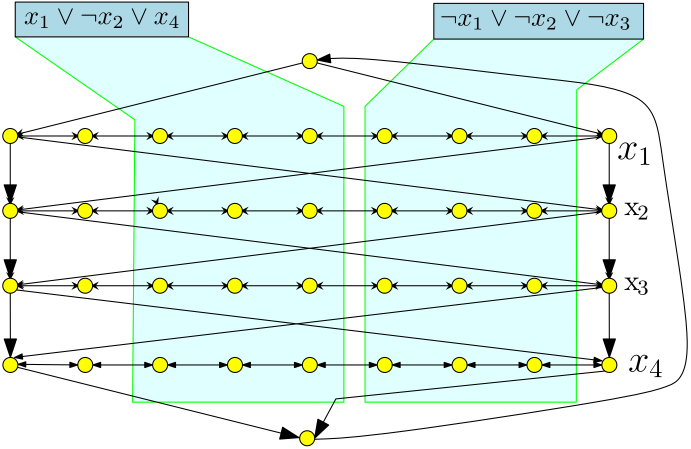

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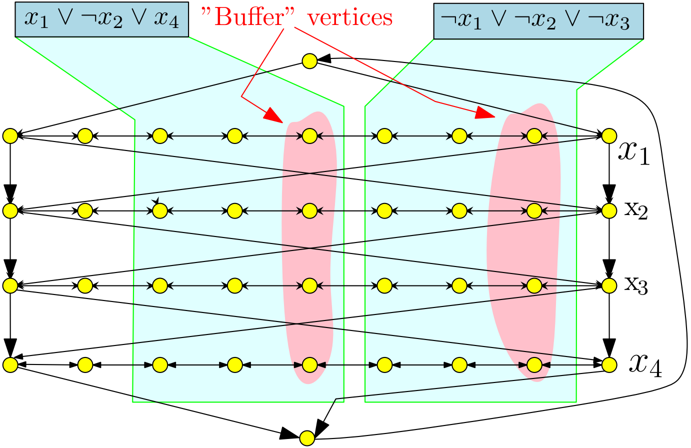

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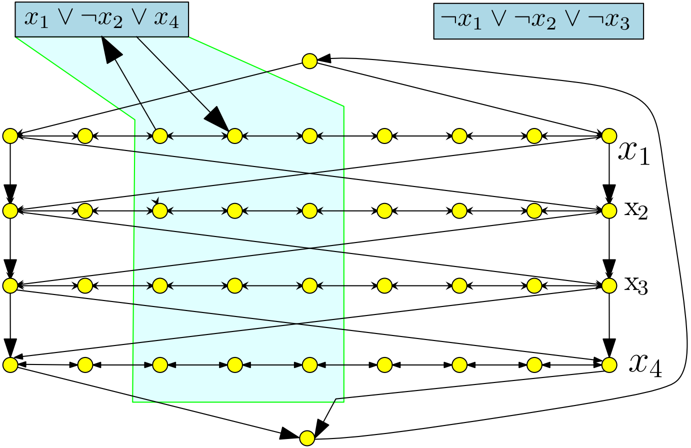

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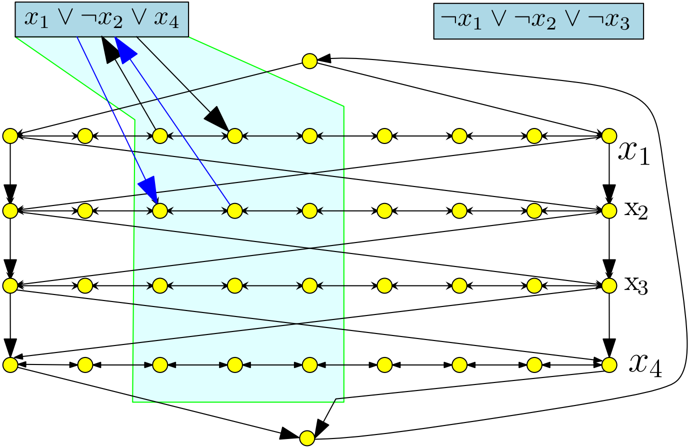

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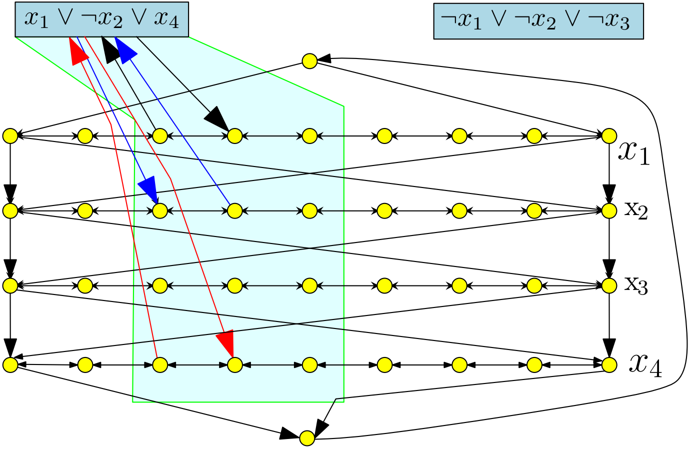

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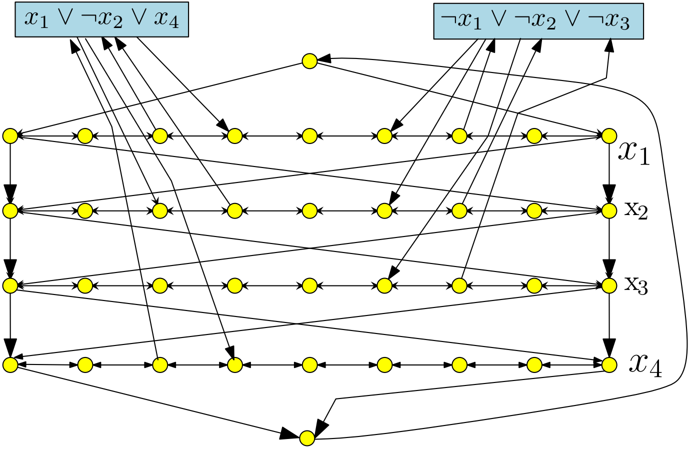

Correctness Proof

Proposition

φ has a satisfying assignment iff G_φ has a Hamiltonian cycle.

Proof.

$\Rightarrow$ Let a be the satisfying assignment for φ. Define Hamiltonian cycle as follows

- If $a(x_i) = 1$ then traverse path i from left to right
- If $a(x_i) = 0$ then traverse path i from right to left
- For each clause, path of at least one variable is in the "right" direction to splice in the node corresponding to clause □

Hamiltonian Cycle $\Rightarrow$ Satisfying assignment

Suppose Π is a Hamiltonian cycle in G_φ

- If Π enters c_j (vertex for clause C_j) from vertex $3j$ on path i then it must leave the clause vertex on edge to $3j+1$ on the *same path* i

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  - Thus, we don't have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle

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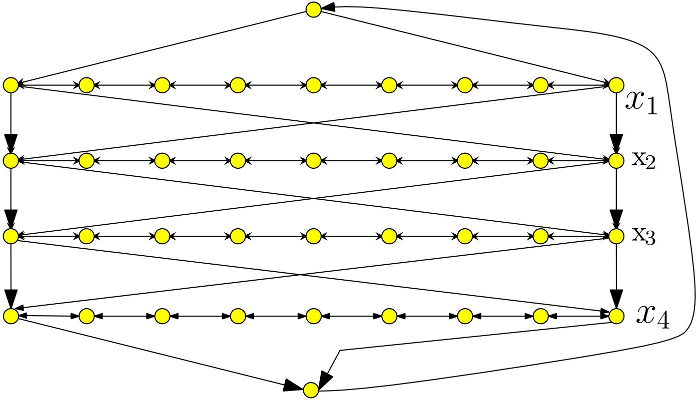
  - Thus, we don't have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle
- Similarly, if Π enters c_j from vertex $3j+1$ on path i then it must leave the clause vertex c_j on edge to $3j$ on path i

Example



Hamiltonian Cycle $\Longrightarrow$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after C_i are connected by an edge
- We can remove c_j from cycle, and get Hamiltonian cycle in $G - c_j$
- Consider Hamiltonian cycle in $G - \{c_1, \ldots c_m\}$; it traverses each path in only one direction, which determines the truth assignment

Undirected Hamiltonian Cycle

Exercises: Show NP-completeness for the following problems.

Input Given undirected graph $G = (V, E)$

Goal Does G have a Hamiltonian cycle? That is, is there a cycle that visits every vertex exactly one (except start and end vertex)?

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Reduce from directed to the un-directed case.

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Modify the reduction we saw from **3-SAT**.

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- A Hamiltonian path is a path in the graph that visits every vertex in G exactly once

Modify the reduction we saw from **3-SAT**.

Also prove that **Hamilton path** in undirected graphs is **NP-Complete**.

Part II

NP-Completeness of Graph Coloring

Graph Coloring

Problem: Graph Coloring

Instance: $G = (V, E)$: Undirected graph, integer k.
Question: Can the vertices of the graph be colored using k colors so that vertices connected by an edge do not get the same color?

Graph 3-Coloring

Problem: 3 Coloring

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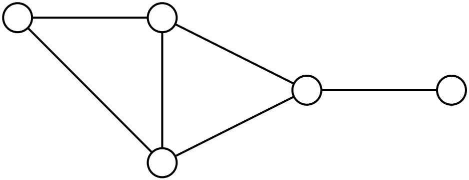

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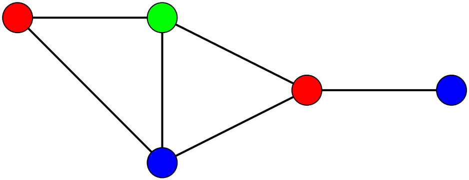

Graph Coloring

Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G. Thus, G can be partitioned into k independent sets iff G is k-colorable.

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Graph 2-Coloring can be decided in polynomial time.

G is 2-colorable iff G is bipartite! There is a linear time algorithm to check if G is bipartite using **BFS**.

Graph Coloring and Register Allocation

Register Allocation

Assign variables to (at most) k registers such that variables needed at the same time are not assigned to the same register

Interference Graph

Vertices are variables, and there is an edge between two vertices, if the two variables are "live" at the same time.

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Observations

- [Chaitin] Register allocation problem is equivalent to coloring the interference graph with k colors
- Moreover, **3-COLOR** $\leq_P$ **k-Register Allocation**, for any $k \geq 3$

Class Room Scheduling

Given n classes and their meeting times, are k rooms sufficient?

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Reduce to Graph k-Coloring problem

Create graph G

- a node v_i for each class i
- an edge between v_i and v_j if classes i and j *conflict*

Class Room Scheduling

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Reduce to Graph k-Coloring problem

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Exercise: G is k-colorable iff k rooms are sufficient

Frequency Assignments in Cellular Networks

Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)

- Breakup a frequency range $[a, b]$ into disjoint *bands* of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
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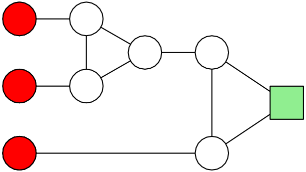
Problem: given k bands and some region with n towers, is there a way to assign the bands to avoid interference?

Can reduce to k-coloring by creating intereference/conflict graph on towers.

3 color this gadget.

You are given three colors: red, green and blue. Can the following graph be three colored in a valid way (assuming the four nodes are already colored as indicated).



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3 color this gadget II

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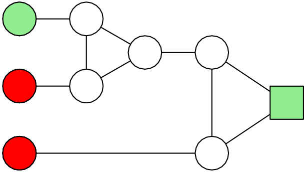

(A) Yes.
(B) No.

3-Coloring is **NP-Complete**

- **3-Coloring** is in **NP**.
  - Certificate: for each node a color from $\{1, 2, 3\}$.
  - Certifier: Check if for each edge (u, v), the color of u is different from that of v.
- Hardness: We will show **3-SAT** $\leq_P$ **3-Coloring**.

Reduction Idea

Start with **3SAT** formula (i.e., $3\mathrm{CNF}$ formula) φ with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$. Create graph G_φ such that G_φ is 3-colorable iff φ is satisfiable

- need to establish truth assignment for $x_1, \ldots, x_n$ via colors for some nodes in G_φ.

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- for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i

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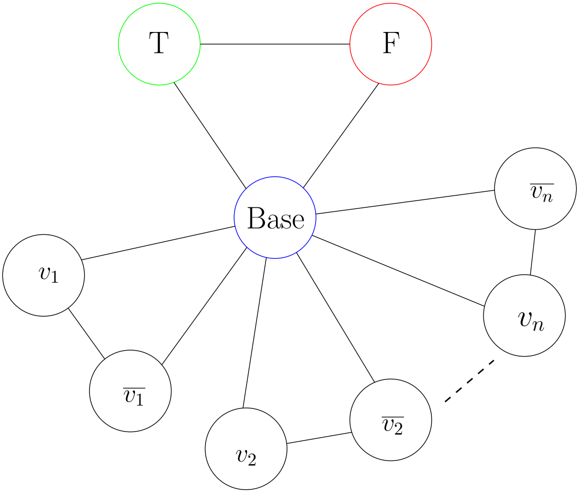
- If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
- Need to add constraints to ensure clauses are satisfied (next phase)

Figure

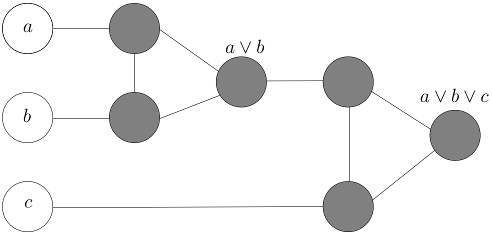


Clause Satisfiability Gadget

For each clause $C_j = (a \vee b \vee c)$, create a small gadget graph

- gadget graph connects to nodes corresponding to a, b, c
- needs to implement OR

OR-gadget-graph:

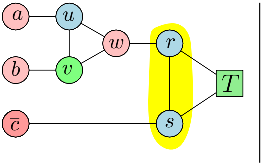


OR-Gadget Graph

Property: if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

Property: if one of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.

3 coloring of the clause gadget



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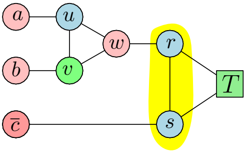

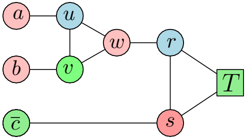

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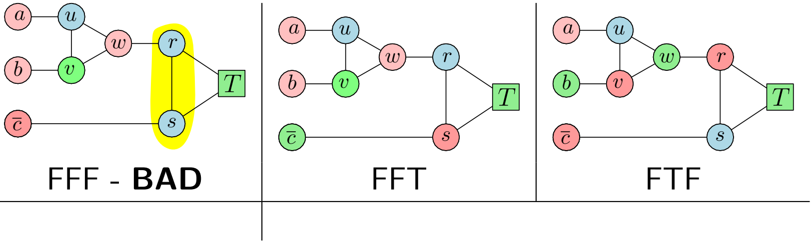
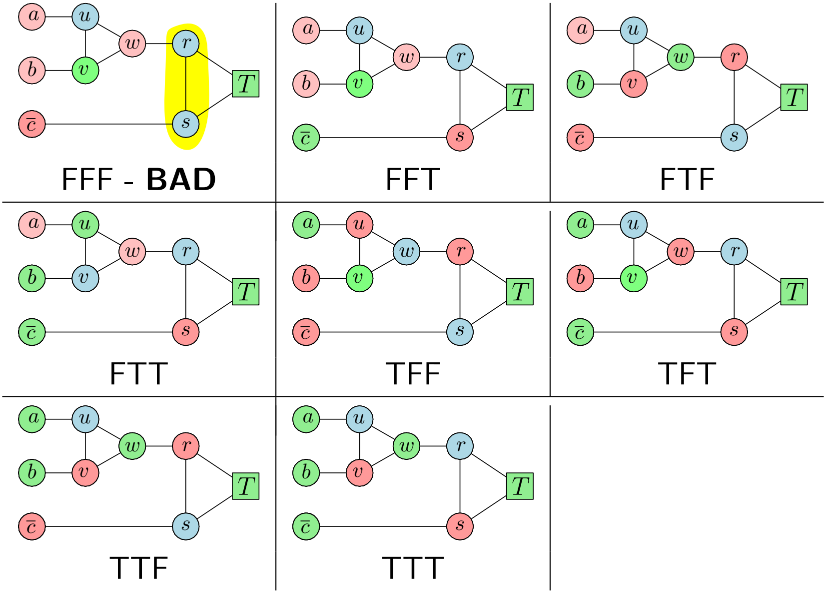


FFF - **BAD** | FFT | FTF


FFF - **BAD**
FFT
FTF
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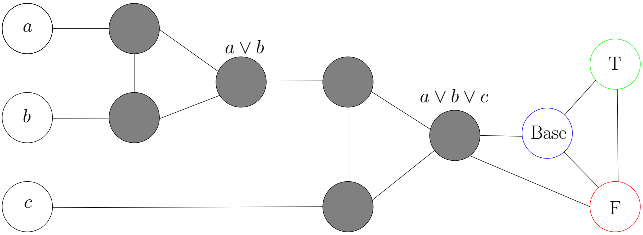
Reduction

- create triangle with nodes True, False, Base
- for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- for each clause $C_j = (a \vee b \vee c)$, add OR-gadget graph with input nodes a, b, c and connect output node of gadget to both False and Base



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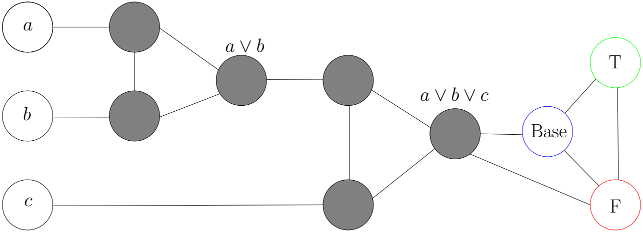


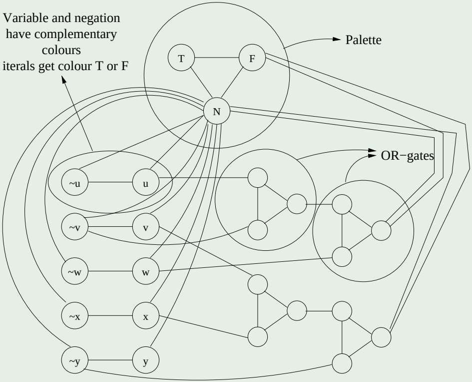
Claim

No legal $\mathbf{3}$*-coloring of above graph (with coloring of nodes* $\boldsymbol{T}, \boldsymbol{F}, \boldsymbol{B}$ *fixed) in which* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *are colored False. If any of* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *are colored True then there is a legal* $\mathbf{3}$*-coloring of above graph.*

Reduction Outline

Example

$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$



Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False

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- for each clause $C_j = (a \vee b \vee c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

Correctness of Reduction

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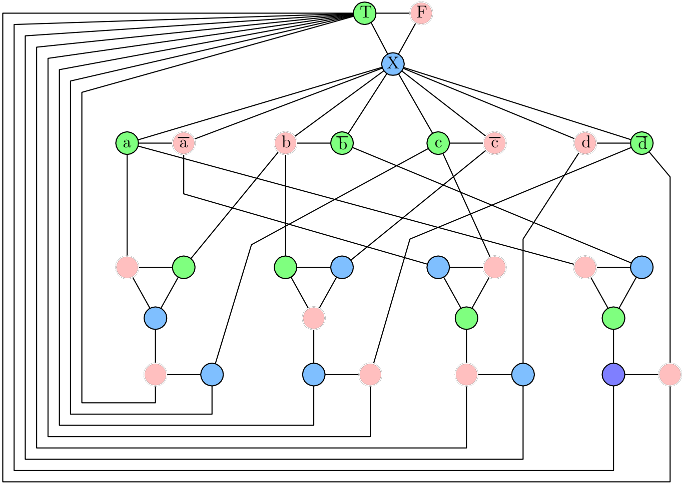
- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Graph generated in reduction...

... from 3SAT to 3COLOR



Need to Know **NP-Complete** Problems

- **SAT** and **3-SAT**
- **Independent Set**
- **Vertex Cover**
- **Clique**
- **Set Cover**
- **Hamiltonian Cycle** in Directed/Undirected Graphs
- **3-Coloring**
- **3-D Matching**
- **Subset Sum** and **Knapsack**