SAT, NP, NP-Completeness

Lecture 21
April 22, 2021

Most slides are courtesy Prof. Chekuri

Part I

The Satisfiability Problem (SAT)

Propositional Formulas

Definition

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   For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.

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   For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
3. A **formula in conjunctive normal form** (CNF) is propositional formula which is a conjunction of clauses
   1. $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.

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4. A formula φ is a $3\mathrm{CNF}$:
   A CNF formula such that every clause has **exactly** 3 literals.
   1. $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a $3\mathrm{CNF}$ formula, but $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

Satisfiability

Problem: SAT

> **Instance:** A CNF formula φ.
> **Question:** Is there a truth assignment to the variable of φ such that φ evaluates to true?

Problem: 3SAT

> **Instance:** A $3\mathrm{CNF}$ formula φ.
> **Question:** Is there a truth assignment to the variable of φ such that φ evaluates to true?

Satisfiability

SAT

Given a CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Example

1. $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is satisfiable; take $x_1, x_2, \ldots x_5$ to be all true
2. $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee x_2)$ is not satisfiable.

Importance of SAT and 3SAT

1. **SAT** and **3SAT** are basic constraint satisfaction problems.
2. Many different problems can reduced to them because of the simple yet powerful expressively of logical constraints.
3. Arise naturally in many applications involving hardware and software verification and correctness.
4. As we will see, it is a fundamental problem in theory of **NP-Complete**ness.

3SAT $\leq_P$ SAT

1. **3SAT** $\leq_P$ **SAT**.
2. Because...
   A **3SAT** instance is also an instance of **SAT**.

SAT $\leq_P$ 3SAT

Claim

SAT $\leq_P$ **3SAT**.

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SAT $\leq_P$ **3SAT**.

Given φ a **SAT** formula we create a **3SAT** formula φ' such that

1. φ is satisfiable iff φ' is satisfiable.
2. φ' can be constructed from φ in time polynomial in $|\varphi|$.

SAT $\leq_P$ 3SAT

How SAT is different from 3SAT?

In **SAT** clauses might have arbitrary length: $\mathbf{1, 2, 3, \ldots}$ variables:

$$\left(x \lor y \lor z \lor w\right) \land \left(\neg x \lor \neg y \lor \neg z \lor w \lor u\right) \land \left(\neg x\right)$$

In **3SAT** every clause must have **exactly 3** different literals.

SAT $\leq_P$ 3SAT

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In **3SAT** every clause must have **exactly 3** different literals.

Consider $\left(\mathbf{x} \vee \mathbf{y} \vee \mathbf{z} \vee \mathbf{w}\right)$

Replace it with $\left(x \vee y \vee \alpha\right) \wedge \left(\neg\alpha \vee w \vee u\right)$

SAT $\leq_P$ 3SAT

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In **3SAT** every clause must have **exactly** 3 different literals.

Consider $\left(x \vee y \vee z \vee w\right)$

Replace it with $\left(x \vee y \vee \alpha\right) \wedge \left(\neg \alpha \vee w \vee u\right)$

1. Pad short clauses so they have 3 literals.
2. Break long clauses into shorter clauses. (Need to add new variables)
3. Repeat the above till we have a $3\mathrm{CNF}$.

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Why the reduction from 3SAT to 2SAT fails?

Consider a clause $(x \vee y \vee z)$. We need to reduce it to a collection of $2\mathrm{CNF}$ clauses. Introduce a face variable α, and rewrite this as

$$(x \vee y \vee \alpha) \wedge (\neg\alpha \vee z) \qquad \text{(bad! clause with 3 vars)}$$

$$\text{or} \quad (x \vee \alpha) \wedge (\neg\alpha \vee y \vee z) \qquad \text{(bad! clause with 3 vars).}$$

(In animal farm language: **2SAT** good, **3SAT** bad.)

What about 2SAT?

A challenging exercise: Given a **2SAT** formula design an efficient algorithm to compute its satisfying assignment...

Look in books etc.

Independent Set

Problem: Independent Set

Instance: A graph G, integer k.
Question: Is there an independent set in G of size k?

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3SAT $\leq_P$ Independent Set

Later (if time permits)

Part II

Definition of P and NP

Problems and Algorithms: Formal Approach

Decision Problems

1. Problem Instance: Binary string s, with size $|s|$
2. Problem: A set X of strings on which the answer should be "yes"; we call these YES instances of X. Strings not in X are NO instances of X.

Definition

1. A is an algorithm for problem X if $A(s) =$ "yes" iff $s \in X$.
2. A is said to have a polynomial running time if there is a polynomial $p(\cdot)$ such that for every string s, $A(s)$ terminates in at most $O(p(|s|))$ steps.

Polynomial Time

Definition

Polynomial time (denoted by **P**) is the class of all (decision) problems that have an algorithm that solves it in polynomial time.

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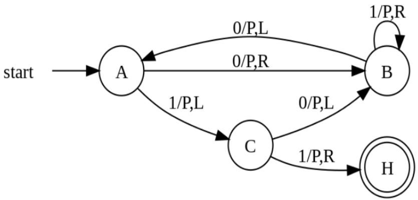
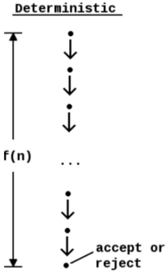
Example

Problems in **P** include

1. Is there a shortest path from s to t of length $\leq k$ in G?
2. Is there a flow of value $\geq k$ in network G?
3. Is there an assignment to variables to satisfy given linear constraints?

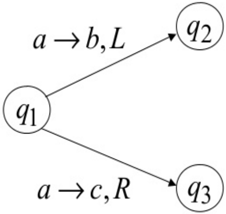
Deterministic Turing Machine



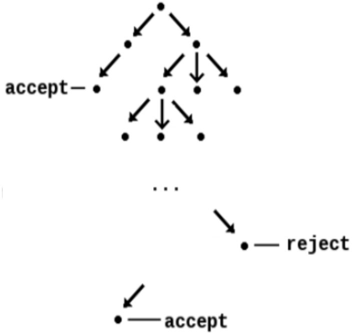


P (polynomial-time): problems that deterministic TM *solves* in polynomial time.

Nondeterministic Turing Machine



Non Deterministic Choice



NP (nondeterministic polynomial time): problems that nondeterministc TM *solves* in polynomial time.

Problems with no known polynomial time algorithms

Problems

1. **Independent Set**
2. **Vertex Cover**
3. **Set Cover**
4. **SAT**
5. **3SAT**

There are of course undecidable problems (no algorithm at all!) but many problems that we want to solve are of similar flavor to the above.

Question: What is common to above problems?

Efficient Checkability

Above problems share the following feature:

Checkability

For any YES instance I_X of X there is a proof/certificate/solution that is of length poly($|I_X|$) such that given a proof one can efficiently check that I_X is indeed a YES instance.

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Examples:

1. **SAT** formula φ: proof is a satisfying assignment.
2. **Independent Set** in graph G and k:

Efficient Checkability

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For any YES instance I_X of X there is a proof/certificate/solution that is of length poly($|I_X|$) such that given a proof one can efficiently check that I_X is indeed a YES instance.

Examples:

1. **SAT** formula φ: proof is a satisfying assignment.
2. **Independent Set** in graph G and k: a subset S of vertices.

Certifiers

Definition (Efficient Certifier.)

An algorithm C is an **efficient certifier** for problem X, if there is a polynomial $p(\cdot)$ such that,

- ⋆ $I_x \in X$ if and only if
  1. there is a string t (certificate/proof) with $|t| \leq p(|I_x|)$,
  2. $C(I_x, t) =$ "yes",
  3. and C runs in polynomial time in $|I_x|$.

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"Guess" the certificate and verify $\Rightarrow$ nondeterministic TM.

Examples

1. Independent set: Does $G = (V, E)$ have an independent set of size $\geq k$?
   1. Certificate: Set $S \subseteq V$.
   2. Certifier: Check $|S| \geq k$ and no pair of vertices in S is connected by an edge.

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3. SAT: Does formula φ have a satisfying truth assignment?

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    1. Certificate: $S \subseteq V$.
    2. Certifier: Check $|S| \leq k$ and that for every edge at least one endpoint is in S.
3. SAT: Does formula φ have a satisfying truth assignment?
    1. Certificate: Assignment a of $0/1$ values to each variable.
    2. Certifier: Check each clause under a and say "yes" if all clauses are true.

Nondeterministic Polynomial Time

Alternate definition

Definition

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Independent Set, **Vertex Cover**, **Set Cover**, **SAT**, **3SAT**, and **Composite** are all examples of problems in **NP**.

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"Guess" the certificate and verify $\Rightarrow$ nondeterministic TM.
nondeterministic TM $\Rightarrow$ Path to an "accept" state is the certificate.

Asymmetry in Definition of NP

Note that only YES instances have a short proof/certificate. NO instances need not have a short certificate.

Example

SAT formula φ. No easy way to prove that φ is NOT satisfiable!

More on this and **co-NP** later on.

P versus NP

Proposition

$\mathbf{P} \subseteq \mathbf{NP}$.

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For a problem in **P** no need for a certificate!

Proof.

Consider problem $X \in \mathbf{P}$ with algorithm A.

1. Certifier C on input I_x, t, runs $A(I_x)$ and returns the answer.
   - C runs in polynomial time.
   - If $I_x \in X$, then for every t, $C(I_x, t) =$ "yes".
   - If $I_x \notin X$, then for every t, $C(I_x, t) =$ "no". □

Exponential Time

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Example: $O(2^n)$, $O(2^{n \log n})$, $O(2^{n^3})$, ...
Problems:

1. **SAT**: try all possible truth assignment to variables.
2. **Independent Set**: try all possible subsets of vertices.
3. **Vertex Cover**: try all possible subsets of vertices.

NP versus EXP

Proposition

NP $\subseteq$ **EXP**.

Proof.

Let $X \in$ **NP** with certifier C. Need to design an exponential time algorithm for X.

1. For every t, with $|t| \leq p(|I_x|)$ run $C(I_x, t)$; answer "yes" if any one of these calls returns "yes".
2. The above algorithm correctly solves X (exercise).
3. Algorithm runs in $O(q(|I_x| + p(|I_x|))2^{p(|I_x|)})$, where q is the running time of C. $\square$

Is **NP** efficiently solvable?

We know $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP}$.

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If $\mathbf{P} = \mathbf{NP}$ this implies that...

(A) **Vertex Cover** can be solved in polynomial time.

(B) $\mathbf{P} = \mathbf{EXP}$.

(C) $\mathbf{EXP} \subseteq \mathbf{P}$.

(D) All of the above.

Is NP efficiently solvable?

We know $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP}$.

Big Question

Is there a problem in **NP** that does not belong to **P**? Or is $\mathbf{P} = \mathbf{NP}$?

Is NP efficiently solvable?

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Big Question

Is there a problem in **NP** that does not belong to **P**? Or is $\mathbf{P} = \mathbf{NP}$?

Status

Relationship between **P** and **NP** remains one of the most important open problems in mathematics/computer science.

Consensus: Most people feel/believe $P \neq NP$.

Resolving **P** versus **NP** is a Clay Millennium Prize Problem. You can win a million dollars in addition to a Turing award and major fame!

If $\mathbf{P} = \mathbf{NP}$ . . .

Or: If pigs could fly then life would be sweet.

1. Many important optimization problems can be solved efficiently.
2. The RSA cryptosystem can be broken.
3. No security on the web.
4. No e-commerce . . .

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Or: If pigs could fly then life would be sweet.

1. Many important optimization problems can be solved efficiently.
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3. No security on the web.
4. No e-commerce ...
5. Creativity can be automated! Proofs for mathematical statement can be found by computers automatically (if short ones exist).

Part III

NP-Completeness and Cook-Levin Theorem

"Hardest" Problems

Question

What is the hardest problem in **NP**? How do we define it?

Towards a definition

1. Hardest problem must be in **NP**.
2. Hardest problem must be at least as "difficult" as every other problem in **NP**.

NP-Complete Problems

Definition

A problem $\boldsymbol{X}$ is said to be **NP-Hard** if

1. (Hardness) $\forall \boldsymbol{Y} \in$ **NP**, we have that **Y** $\leq_{\boldsymbol{P}}$ **X**.

NP-Complete Problems

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An **NP-Hard** problem need not be in **NP**!

Example: Halting problem is **NP-Hard** (why?) but not **NP-Complete**.

Solving NP-Complete Problems

Proposition

*Suppose X is **NP-Complete**. Then X can be solved in polynomial time if and only if $\mathbf{P} = \mathbf{NP}$.*

Proof.

$\Rightarrow$ Suppose X can be solved in polynomial time

1. Let $Y \in \mathbf{NP}$. We know $\mathbf{Y} \leq_P \mathbf{X}$.
2. Then Y can be solved in polynomial time. $Y \in \mathbf{P}$.

Solving **NP-Complete** Problems

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Suppose X is **NP-Complete***. Then X can be solved in polynomial time if and only if* $\mathsf{P} = \mathsf{NP}$.

Proof.

$\Rightarrow$ Suppose X can be solved in polynomial time

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2. Then Y can be solved in polynomial time. $Y \in \mathsf{P}$.
3. Thus, $Y \in \mathsf{NP} \Rightarrow Y \in P$; $NP \subseteq P$.
4. Since $\mathsf{P} \subseteq \mathsf{NP}$, we have $\mathsf{P} = \mathsf{NP}$.

Solving **NP-Complete** Problems

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3. Thus, $Y \in \mathbf{NP} \Rightarrow Y \in P$; $NP \subseteq P$.
4. Since $\mathbf{P} \subseteq \mathbf{NP}$, we have $\mathbf{P} = \mathbf{NP}$.

$\Leftarrow$ Since $\mathbf{P} = \mathbf{NP}$, and $X \in \mathbf{NP}$, we have a polynomial time algorithm for X. $\square$

Consequences of proving NP-Completeness

If X is **NP-Complete**

1. Since we believe $\mathbf{P} \neq \mathbf{NP}$,
2. and solving X efficiently implies $\mathbf{P} = \mathbf{NP}$.

X is unlikely to be efficiently solvable.

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(This is proof by mob opinion — take with a grain of salt.)

NP-Complete Problems

Question

Are there any problems that are **NP-Complete**?

Answer

Yes! Many, many problems are **NP-Complete**.

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Cook-Levin Theorem:

Theorem

SAT *is* ***NP-Complete****.*

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Cook-Levin Theorem:

Theorem

SAT *is* **NP-Complete**.

Using reductions one can prove that many other problems are **NP-Complete**

Proving that a problem X is NP-Complete

To prove X is **NP-Complete**, show

1. Show X is in **NP**.
   1. certificate/proof of polynomial size in input
   2. polynomial time certifier $C(s, t)$
2. Reduction from a known **NP-Complete** problem such as **3SAT** or **SAT** to X

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SAT $\leq_P$ X implies that every **NP** problem $Y \leq_P X$. Why?

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SAT $\leq_P$ X implies that every **NP** problem $Y \leq_P X$. Why? Transitivity of reductions:

$Y \leq_P SAT$ and $SAT \leq_P X$ and hence $Y \leq_P X$.

Recap . . .

Problems

1. **Independent Set**
2. **Clique**
3. **Vertex Cover**
4. **Set Cover**
5. **SAT**
6. **3SAT**

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Relationship

3SAT $\leq_P$ **Independent Set**

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Relationship

3SAT $\leq_P$ **Independent Set** $\overset{\leq_P}{\geq_P}$ **Clique** $\overset{\leq_P}{\geq_P}$ **Vertex Cover**

Recap ...

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Relationship

3SAT $\leq_P$ **Independent Set** $\underset{\geq_P}{\leq_P}$ **Clique** $\underset{\geq_P}{\leq_P}$ **Vertex Cover** $\leq_P$ **Set Cover**

Recap . . .

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Relationship

$$\textbf{3SAT} \le_P \textbf{Independent Set} \begin{array}{c}\le_P\\ \ge_P\end{array} \textbf{Clique} \begin{array}{c}\le_P\\ \ge_P\end{array} \textbf{Vertex Cover} \le_P \textbf{Set Cover}$$

$$\textbf{3SAT} \le_P \textbf{SAT} \le_P \textbf{3SAT}$$

NP-Completeness via Reductions

1. **SAT** is **NP-Complete**.
2. **SAT** $\leq_P$ **3-SAT** and hence 3-SAT is **NP-Complete**.
3. **3-SAT** $\leq_P$ **Independent Set** (which is in **NP**) and hence **Independent Set** is **NP-Complete**.
4. **Clique** is **NP-Complete**
5. **Vertex Cover** is **NP-Complete**
6. **Set Cover** is **NP-Complete**
7. **Hamilton Cycle** is **NP-Complete**
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Hundreds and thousands of different problems from many areas of science and engineering have been shown to be **NP-Complete**.

A surprisingly frequent phenomenon!

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The reduction 3SAT $\leq_P$ Independent Set

Input: Given a $3\mathrm{CNF}$ formula φ

Goal: Construct a graph G_φ and number k such that G_φ has an independent set of size k if and only if φ is satisfiable.

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Importance of reduction: Although **3SAT** is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.

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2. Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of **3SAT** to construct the reduction.

The Reduction

1. G_φ will have one vertex for each literal in a clause

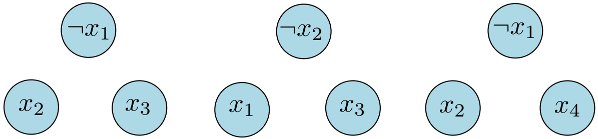

Figure: Graph for
$\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

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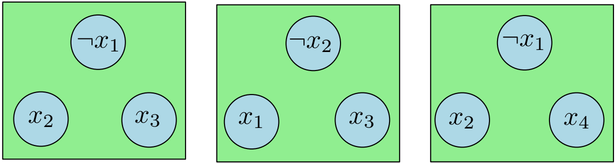

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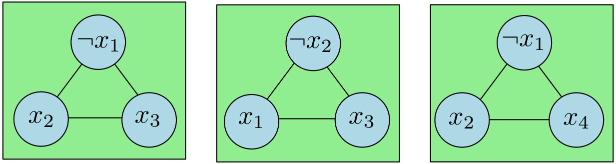

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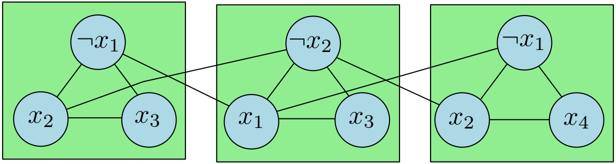

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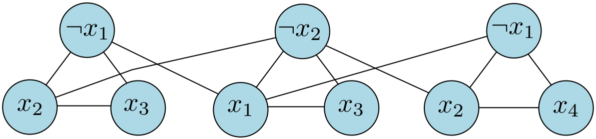
4. Take k to be the number of clauses



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Proposition

φ is satisfiable iff G_φ has an independent set of size k ($=$ number of clauses in φ).

Proof.

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Proof.

$\Rightarrow$ Let a be the truth assignment satisfying φ

1. Pick one of the vertices, corresponding to true literals under a, from each triangle. This is an independent set of the appropriate size □

Correctness (contd)

Proposition

φ is satisfiable iff G_φ has an independent set of size k ($=$ number of clauses in φ).

Proof.

$\Leftarrow$ Let S be an independent set of size k

1. S must contain exactly one vertex from each clause
2. S cannot contain vertices labeled by conflicting clauses
3. Thus, it is possible to obtain a truth assignment that makes in the literals in S true; such an assignment satisfies one literal in every clause $\square$

Part IV

co-NP

NP vs co-NP

NP: Problems with polynomial time verifier for a "yes" instance.

SAT: Given a CNF formula ϕ, does there exists a satisfying assignment?
- Poly-time verification (proof) for "yes" instances.

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co-NP: Complements of decision problems in NP.

- **No-Independent-Set**, **Is-Prime**, **No-Clique**..
- Poly-time verification for "no" instances
- "no" instances can be solved in non-deterministic polynomial time.

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Given integers q and n, is there a prime factor of q larger than n?

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Int-Factorization $\in$ **NP** $\cap$ **co-NP**.

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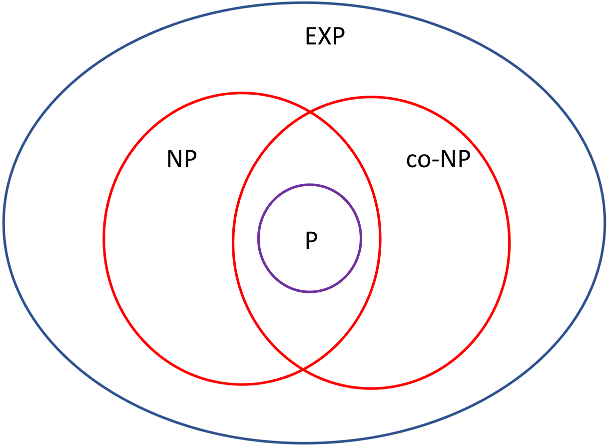
Int-Factorization $\in$ **NP** $\cap$ **co-NP**. But not known to be in **P**.

Landscape of Containment



Part V

Hardness of **Subset Sum**

Subset Sum

Problem: Subset Sum

Instance: S, set of positive integers; t, an integer number (Target)
Question: Is there a subset $X \subseteq S$ such that $\sum_{x \in X} x = t$?

Claim

Subset Sum *is* **NP-Complete**.

Vec Subset Sum

We will prove following problem is **NP-Complete**...

Problem: Vec Subset Sum

Instance: S, set of n vectors of dimension k, each vector has non-negative numbers for its coordinates, and a target vector $\overrightarrow{t}$.
Question: Is there a subset $X \subseteq S$ such that $\sum_{\overrightarrow{x} \in X} \overrightarrow{x} = \overrightarrow{t}$?

Reduction from **3SAT**.

Vec Subset Sum

Handling a single clause

Think about vectors as being lines in a table.

How to “select” exactly one of $x = 0$ and $x = 1$.

First gadget

Selecting between two lines.

Target	??	??	01	???
a_1	??	??	01	??
a_2	??	??	01	??

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a_2	??	??	01	??

Two rows for every variable x: selecting either $x = 0$ or $x = 1$.

Handling a clause...

We will have a column for every clause...

numbers	...	$C \equiv a \vee b \vee \overline{c}$	...
a	...	01	...
$\overline{a}$	...	00	...
b	...	01	...
$\overline{b}$	...	00	...
c	...	00	...
$\overline{c}$	...	01	...
C fix-up 1	000	07	000
C fix-up 2	000	08	000
C fix-up 3	000	09	000
TARGET		10	

3SAT to Vec Subset Sum

numbers	$a \vee \overline{a}$	$b \vee \overline{b}$	$c \vee \overline{c}$	$d \vee \overline{d}$	$D \equiv \overline{b} \vee c \vee \overline{d}$	$C \equiv a \vee b \vee \overline{c}$
a	1	0	0	0	00	01
$\overline{a}$	1	0	0	0	00	00
b	0	1	0	0	00	01
$\overline{b}$	0	1	0	0	01	00
c	0	0	1	0	01	00
$\overline{c}$	0	0	1	0	00	01
d	0	0	0	1	00	00
$\overline{d}$	0	0	0	1	01	01
C fix-up 1	0	0	0	0	00	07
C fix-up 2	0	0	0	0	00	08
C fix-up 3	0	0	0	0	00	09
D fix-up 1	0	0	0	0	07	00
D fix-up 2	0	0	0	0	08	00
D fix-up 3	0	0	0	0	09	00
TARGET	1	1	1	1	10	10

Vec Subset Sum to Subset Sum

numbers
010000000001
010000000000
000100000001
000100000100
000001000100
000001000001
000000010000
000000010101
000000000007
000000000008
000000000009
000000000700
000000000800
000000000900
010101011010

Subset Sum: Weak vs Strong NP-completeness

Subset Sum can be solved in $O(nB)$ time using dynamic programming (exercise).

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Implies that problem is hard only when numbers $a_1, a_2, \ldots, a_n$ are exponentially large compared to n. That is, each a_i requires polynomial in n bits.
Number problems of the above type are said to be **weakly NP-Complete**.

Number problems which are **NP-Complete** even when the numbers are written in unary are **strongly NP-Complete**.

A Strongly NP-Complete Number Problem

3-Partition: Given $3n$ numbers $a_1, a_2, \ldots, a_{3n}$ and target B can the numbers be partitioned into n groups of 3 each such that the sum of numbers in each group is exactly B?

Can further assume that each number a_i is between $B/3$ and $2B/3$.

Can reduce **3-D-Matching** to **3-Partition** in polynomial time such that each number a_i can be written in unary.

Other NP-Complete Problems

- **Hamiltonian cycle**
- **Graph coloring**
- **3-Dimensional Matching**
- **3-Partition**
- ...

Read book.