CS 473: Algorithms

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Introduction to Linear Programming

Lecture 17

March 30, 2021

Some of the slides are courtesy Prof. Chekuri

Part I

Introduction to Linear Programming

A Factory Example

Problem

Your factory can produce Laptop and iPhone using Copper.

1. One ton of Copper $\longrightarrow$ one Laptop
2. One ton of Copper $\longrightarrow$ one iPhone
3. We have 200 tons of Copper.
4. Laptop can be sold for **\$1** and iPhone for **\$6**.

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Solution:

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How many units of Laptop and iPhone should your factory manufacture to maximize profit?

Solution: manufacture only iPhone

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Your factory can produce Laptop and iPhone using resources C, B, A.

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1. One unit of A and C each $\longrightarrow$ One Laptop
2. One unit of B and C each $\longrightarrow$ One iPhone
3. We have 200 units of A, 300 units of B, and 400 units of C.
4. Product Laptop can be sold for **\$1** and product iPhone for **\$6**.

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Solution: Formulate as a linear program.

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How many units to manufacture to max profit?

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Suppose x_1 units of Laptop and x_2 units of iPhone.

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How many units to manufacture to max profit?

Suppose x_1 units of Laptop and x_2 units of iPhone.

$$
\begin{array}{lll}
\textbf{max} & x_1 + 6x_2 & \\
\text{s.t.} & x_1 \leq 200 & (A) \\
& x_2 \leq 300 & (B) \\
& x_1 + x_2 \leq 400 & (C) \\
& x_1 \geq 0 & \\
& x_2 \geq 0 &
\end{array}
$$

Linear Programming Formulation

Let us produce x_1 units of Laptop and x_2 units of iPhone. Our profit can be computed by solving

$$\begin{array}{ll} \text{maximize} & x_1 + 6x_2 \\ \text{subject to} & x_1 \leq 200 \quad x_2 \leq 300 \quad x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{array}$$

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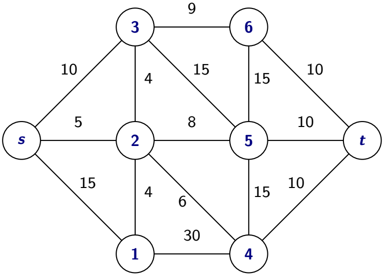
What is the solution?

Maximum Flow in Network



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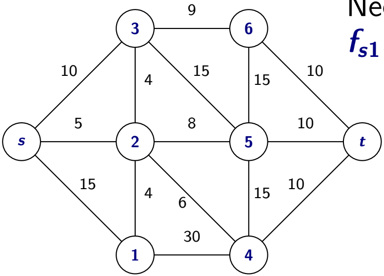

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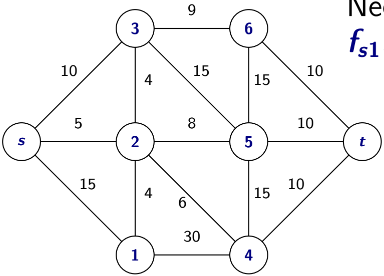

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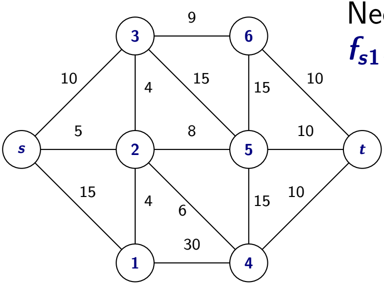
Maximum Flow in Network



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and

$$
\begin{array}{lll}
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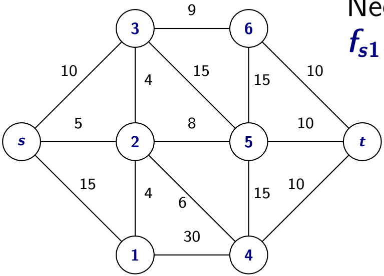

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$$

$$
f_{s1} \geq 0 \quad f_{s2} \geq 0 \quad f_{s3} \geq 0 \quad \cdots \quad f_{4t} \geq 0 \quad f_{5t} \geq 0 \quad f_{6t} \geq 0
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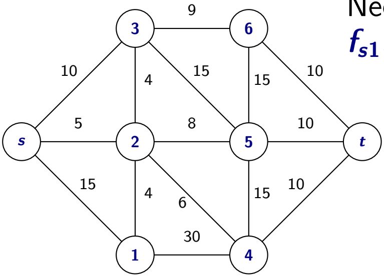
Maximum Flow in Network



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$$
f_{s1} \ge 0 \quad f_{s2} \ge 0 \quad f_{s3} \ge 0 \quad \cdots \quad f_{4t} \ge 0 \quad f_{5t} \ge 0 \quad f_{6t} \ge 0
$$

maximize: $f_{s1} + f_{s2} + f_{s3}$.

Maximum Flow as a Linear Program

For a general flow network $G = (V, E)$ with capacities c_e on edge $e \in E$, we have variables f_e indicating flow on edge e

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{e \text{ out of } s} f_e & \\
\text{subject to} \quad & f_e \leq c_e & \text{for each } e \in E \\
& \sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e = 0 & \forall v \in V \setminus \{s, t\} \\
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& f_e \geq 0 & \text{for each } e \in E.
\end{aligned}
$$

Number of variables: m, one for each edge.
Number of constraints: $m + n - 2 + m$.

Minimum Cost Flow with Lower Bounds

... as a Linear Program

For a general flow network $G = (V, E)$ with capacities c_e, lower bounds ℓ_e, and costs w_e, we have variables f_e indicating flow on edge e. Suppose we want a min-cost flow of value at least F.

$$\text{Minimize} \sum_{e \in E} w_e f_e$$

$$\text{subject to} \sum_{e \text{ out of } s} f_e \geq F$$

$$f_e \leq c_e \qquad f_e \geq \ell_e \qquad \text{for each } e \in E$$

$$\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e = 0 \quad \text{for each } v \in V - \{s, t\}$$

$$f_e \geq 0 \qquad \text{for each } e \in E.$$

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& f_e \geq 0 && \text{for each } e \in E.
\end{aligned}
$$

Number of variables: m, one for each edge

Number of constraints: $1 + m + m + n - 2 + m = 3m + n - 1$.

Linear Programs

Problem

Find a vector $x \in \mathbb{R}^d$ that

$$\begin{array}{lll} \text{maximize/minimize} & \sum_{j=1}^{d} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i & \text{for } i = 1 \dots p \\ & \sum_{j=1}^{d} a_{ij} x_j = b_i & \text{for } i = p+1 \dots q \\ & \sum_{j=1}^{d} a_{ij} x_j \geq b_i & \text{for } i = q+1 \dots n \end{array}$$

Linear Programs

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 & \sum_{j=1}^{d} a_{ij} x_j \geq b_i & \text{for } i = q+1 \ldots n
\end{array}
$$

Input is matrix $A = (a_{ij}) \in \mathbb{R}^{n \times d}$, column vector $b = (b_i) \in \mathbb{R}^n$, and row vector $c = (c_j) \in \mathbb{R}^d$

Canonical Form of Linear Programs

Canonical Form

A linear program is in canonical form if it has the following structure

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for } i = 1 \ldots n \end{array}$$

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Conversion to Canonical Form

1. Replace $\sum_j a_{ij} x_j = b_i$ by

$$\sum_j a_{ij} x_j \leq b_i \quad \text{and} \quad -\sum_j a_{ij} x_j \leq -b_i$$

2. Replace $\sum_j a_{ij} x_j \geq b_i$ by $-\sum_j a_{ij} x_j \leq -b_i$

Matrix Representation of Linear Programs

A linear program in canonical form can be written as

$$\begin{array}{ll} \text{maximize} & \boldsymbol{c \cdot x} \\ \text{subject to} & \boldsymbol{Ax \leq b} \end{array}$$

where $\boldsymbol{A = (a_{ij})} \in \mathbb{R}^{\boldsymbol{n \times d}}$, column vector $\boldsymbol{b = (b_i)} \in \mathbb{R}^{\boldsymbol{n}}$, row vector $\boldsymbol{c = (c_j)} \in \mathbb{R}^{\boldsymbol{d}}$, and column vector $\boldsymbol{x = (x_j)} \in \mathbb{R}^{\boldsymbol{d}}$

1. Number of variable is $\boldsymbol{d}$
2. Number of constraints is $\boldsymbol{n}$

Other Standard Forms for Linear Programs

$$\begin{array}{ll} \text{maximize} & \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0} \end{array}$$

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Linear Programming: A History

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3. First algorithm (Simplex) to solve linear programs by George Dantzig in 1947
4. Kantorovich and Koopmans receive Nobel Prize for economics in 1975 ; Dantzig, however, was ignored
    1. Koopmans contemplated refusing the Nobel Prize to protest Dantzig's exclusion, but Kantorovich saw it as a vindication for using mathematics in economics, which had been written off as "a means for apologists of capitalism"