CS 473: Algorithms

Ruta Mehta

University of Illinois, Urbana-Champaign

Spring 2021
Inequalities & Randomized QuickSort

Lecture 8
Feb 18, 2021

Most slides are courtesy Prof. Chekuri
Outline

Slick Analysis of Randomized **QuickSort**

**Concentration of Mass Around Mean**
Markov’s Inequality

Chebyshev’s Inequality

Chernoff Bound

Randomized **QuickSort**: High Probability Analysis
Part I

Analysis of QuickSort
Recall: Randomized **QuickSort**

**Randomized QuickSort**

1. Pick a pivot element *uniformly at random* from the array.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

**Theorem**

*Expected running time of Randomized QuickSort on an array of size* $n$ *is* $O(n \log n)$.
**Analysis via Recurrence**

1. **A**: Given array with \( n \) distinct numbers.

2. **\( Q(A) \)**: number of comparisons of randomized **QuickSort** on \( A \). Note that \( Q(A) \) is a random variable.

3. **\( X_i \)**: Random variable indicating if picked pivot has rank \( i \) in \( A \).

\( A^i_{\text{left}} \) and \( A^i_{\text{right}} \) be the corresponding left and right subarrays.

\[
Q(A) = n + \sum_{i=1}^{n} X_i \cdot \left( Q(A^i_{\text{left}}) + Q(A^i_{\text{right}}) \right).
\]

Exactly one non-zero \( X_i \). \( E[X_i] = \Pr[\text{pivot has rank } i] = 1/n. \)
Independence of Random Variables

Lemma

Random variables $X_i$ is independent of random variables $Q(A^i_{\text{left}})$ as well as $Q(A^i_{\text{right}})$, i.e.

$$E[X_i \cdot Q(A^i_{\text{left}})] = E[X_i] E\left[Q(A^i_{\text{left}})\right]$$

$$E[X_i \cdot Q(A^i_{\text{right}})] = E[X_i] E\left[Q(A^i_{\text{right}})\right]$$

Proof.

This is because the algorithm, while recursing on $Q(A^i_{\text{left}})$ and $Q(A^i_{\text{right}})$ uses new random coin tosses that are independent of the coin tosses used to decide the first pivot. Only the latter decides value of $X_i$.
$T(n) = \max_{A : |A| = n} E[Q(A)]$ be the worst-case expected running time on arrays of size $n$.

We have, for any $A$:

$$Q(A) = n + \sum_{i=1}^{n} X_i \left( Q(A^i_{\text{left}}) + Q(A^i_{\text{right}}) \right)$$
Analysis via Recurrence

\[ T(n) = \max_{A: |A| = n} \mathbb{E}[Q(A)] \] be the worst-case expected running time on arrays of size \( n \).

We have, for any \( A \):

\[
Q(A) = n + \sum_{i=1}^{n} X_i \left( Q(A_{\text{left}}^i) + Q(A_{\text{right}}^i) \right)
\]

By linearity of expectation, and independence random variables:

\[
\mathbb{E}[Q(A)] = n + \sum_{i=1}^{n} \mathbb{E}[X_i] \left( \mathbb{E}[Q(A_{\text{left}}^i)] + \mathbb{E}[Q(A_{\text{right}}^i)] \right)
\]

\[
\leq n + \sum_{i=1}^{n} \frac{1}{n} \left( T(i - 1) + T(n - i) \right)
\]
Analysis via Recurrence

\[ T(n) = \max_{A:|A|=n} E[Q(A)] \] be the worst-case expected running time on arrays of size \( n \).

We derived:

\[
E\left[ Q(A) \right] \leq n + \sum_{i=1}^{n} \frac{1}{n} (T(i - 1) + T(n - i)) .
\]

Note that above holds for any \( A \) of size \( n \). Therefore

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Solving the Recurrence

\[ T(n) \leq n + \sum_{i=1}^{n} \frac{1}{n} (T(i - 1) + T(n - i)) \]

with base case \( T(1) = 0 \).

**Lemma**

\[ T(n) = O(n \log n) \]
Solving the Recurrence

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**Lemma**

\[ T(n) = O(n \log n) \]

**Proof.**

(Guess and) Verify by induction.
Part II

Slick analysis of QuickSort
A Slick Analysis of QuickSort

\[ Q(A) : \text{number of comparisons done on input array } A \]

1. Rank of an element is its position in the sorted \( A \).
2. \( R_{ij} \) : event that rank \( i \) element is compared with rank \( j \) element, for \( 1 \leq i < j < n \).
A Slick Analysis of QuickSort

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3. \( X_{ij} : \) the indicator random variable for \( R_{ij} \). That is, \( X_{ij} = 1 \) if rank \( i \) is compared with rank \( j \) element, otherwise \( 0 \).
A Slick Analysis of QuickSort

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3. \( X_{ij} \): the indicator random variable for \( R_{ij} \). That is, \( X_{ij} = 1 \) if rank \( i \) is compared with rank \( j \) element, otherwise 0.

\[
Q(A) = \sum_{1 \leq i < j \leq n} X_{ij}
\]

and hence by linearity of expectation,

\[
E\left[ Q(A) \right] = \sum_{1 \leq i < j \leq n} E\left[ X_{ij} \right] = \sum_{1 \leq i < j \leq n} \Pr \left[ R_{ij} \right].
\]
$R_{ij} = \text{rank } i \text{ element is compared with rank } j \text{ element.}$

**Question:** What is $\Pr[R_{ij}]$?
A Slick Analysis of **QuickSort**

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**Question:** What is \( \Pr[R_{ij}] \)?

With ranks: 6 4 8 1 2 3 7 5
A Slick Analysis of QuickSort

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**Question:** What is \( \text{Pr}[R_{ij}] \)?

\[
\begin{array}{cccccccc}
7 & 5 & 9 & 1 & 3 & 4 & 8 & 6 \\
\end{array}
\]

With ranks: 6 4 8 1 2 3 7 5

As such, probability of comparing 5 to 8 is \( \text{Pr}[R_{4,7}] \).
A Slick Analysis of QuickSort

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1. If pivot too small (say 3 [rank 2]). Partition and call recursively:

Decision if to compare 5 to 8 is moved to subproblem.
A Slick Analysis of QuickSort

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   \end{array}
   \Rightarrow \begin{array}{cccccccc}
   1 & 3 & 7 & 5 & 9 & 4 & 8 & 6 \\
   \end{array}
   \]

   Decision if to compare 5 to 8 is moved to subproblem.

2. If pivot too large (say 9 [rank 8]):

   \[
   \begin{array}{cccccccc}
   7 & 5 & 9 & 1 & 3 & 4 & 8 & 6 \\
   \end{array}
   \Rightarrow \begin{array}{cccccccc}
   7 & 5 & 1 & 3 & 4 & 8 & 6 & 9 \\
   \end{array}
   \]

   Decision if to compare 5 to 8 moved to subproblem.
A Slick Analysis of **QuickSort**

**Question:** What is $Pr[R_{i,j}]$?

As such, probability of comparing 5 to 8 is $Pr[R_{4,7}]$.

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2. If pivot is 8 (rank 7). Bingo!
A Slick Analysis of QuickSort

**Question:** What is \( \text{Pr}[R_{i,j}] \)?

<table>
<thead>
<tr>
<th>7 5 9 1 3 4 8 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 4 8 1 2 3 7 5</td>
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</table>

As such, probability of comparing 5 to 8 is \( \text{Pr}[R_{4,7}] \).

1. **If pivot is 5 (rank 4). Bingo!**

   ![Image 1](attachment:image1.png)

2. **If pivot is 8 (rank 7). Bingo!**

   ![Image 2](attachment:image2.png)

3. **If pivot in between the two numbers (say 6 [rank 5]):**

   ![Image 3](attachment:image3.png)

   5 and 8 will never be compared to each other.
Question: What is $Pr[R_{i,j}]$?

Conclusion:

$R_{i,j}$ happens if and only if:

- $i$th or $j$th ranked element is the first pivot out of $i$th to $j$th ranked elements.

$Pr[R_{i,j}] = Pr[i$th or $j$th ranked element is the pivot $|$ pivot has rank in $\{i, i+1 \ldots, j-1, j\}]$
A Slick Analysis of QuickSort

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There are $k = j - i + 1$ relevant elements.
A Slick Analysis of **QuickSort**

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$$\Pr[R_{i,j}] = \Pr[i\text{th or } j\text{th ranked element is the pivot} \mid \text{pivot has rank in } \{i, i+1 \ldots, j-1, j\}]$$

There are $k = j - i + 1$ relevant elements.

$$\Pr[R_{i,j}] = \frac{2}{k} = \frac{2}{j - i + 1}.$$
Question: What is $\Pr[R_{ij}]$?

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$
A Slick Analysis of QuickSort

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**Lemma**

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**Proof.**

Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be elements of $A$ in sorted order. Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$
A Slick Analysis of QuickSort

**Question:** What is $\Pr[R_{ij}]$?

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**Observation:** If pivot is chosen outside $S$ then all of $S$ either in left array or right array.
A Slick Analysis of QuickSort

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Observation: $a_i$ is compared with $a_j$ if and only if either $a_i$ or $a_j$ is chosen as a pivot from $S$ at separation...
A Slick Analysis of *QuickSort*

Continued...

**Lemma**

\[
\Pr[R_{ij}] = \frac{2}{j - i + 1}.
\]

**Proof.**

Let \(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n\) be sort of \(A\). Let \(S = \{a_i, a_{i+1}, \ldots, a_j\}\)

**Observation:** \(a_i\) is compared with \(a_j\) if and only if either \(a_i\) or \(a_j\) is chosen as a pivot from \(S\) at separation.

**Observation:** Given that pivot is chosen from \(S\) the probability that it is \(a_i\) or \(a_j\) is exactly \(2/|S| = 2/(j - i + 1)\) since the pivot is chosen uniformly at random from the array.
A Slick Analysis of QuickSort

Continued...

\[ E\left[ Q(A) \right] = \sum_{1 \leq i < j \leq n} E[X_{ij}] = \sum_{1 \leq i < j \leq n} \text{Pr}[R_{ij}]. \]

Lemma

\[ \text{Pr}[R_{ij}] = \frac{2}{j - i + 1}. \]
A Slick Analysis of QuickSort

Continued...

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\[
Pr[R_{ij}] = \frac{2}{j-i+1}.
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\[
E\left[ Q(A) \right] = \sum_{1 \leq i < j \leq n} \frac{2}{j-i+1} \\
= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}
\]
A Slick Analysis of \textit{QuickSort}

Continued...

\textbf{Lemma}

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$ 

$$\mathbb{E}[Q(A)] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}.$$
A Slick Analysis of QuickSort

Continued...

**Lemma**

\[
\Pr[R_{ij}] = \frac{2}{j-i+1}.
\]

\[
\mathbb{E}[Q(A)] = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j-i+1}
\]
Lemma

\[ \Pr[R_{ij}] = \frac{2}{j - i + 1}. \]

\[
\mathbb{E}[Q(A)] = 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j - i + 1} = 2 \sum_{i=1}^{n-1} \sum_{\Delta=2}^{n-i+1} \frac{1}{\Delta}
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= 2 \sum_{i=1}^{n-1} (H_{n-i+1} - 1) \leq 2 \sum_{1 \leq i < n} H_n
\]

\[ H_k = \sum_{i=1}^{k} \frac{1}{i} = \Theta(\log k) \]
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\]

\[
\leq 2nH_{n} = O(n \log n)
\]

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Part III

Inequalities
Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: $k$ w.p. $\binom{n}{k}1/2^n$. 
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![Graph showing probability distribution for $n=4$.]
Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: $k$ w.p. $\binom{n}{k} \frac{1}{2^n}$. 

![Graph showing the binomial distribution with $n = 8$. The x-axis represents the number of 1s from 0 to 8, and the y-axis represents the probability from 0 to 0.3. The graph peaks at $n = 4$.](image)
Massive randomness. Is not that random.

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Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: $k$ w.p. $\binom{n}{k} \frac{1}{2^n}$. 

![Graph showing the binomial distribution for $n = 512$.]
Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: $k$ w.p. $\binom{n}{k}/2^n$. 

![Graph showing probability distribution for flipping a coin $n$ times, with $n = 1024$.]
Massive randomness.. Is not that random.

Consider flipping a fair coin $n$ times independently, head gives $1$, tail gives zero. How many $1$s? Binomial distribution: $k$ w.p. $\binom{n}{k} \frac{1}{2^n}$. 

![Graph showing probability distribution for $n = 2048$.]
Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: $k$ w.p. $\binom{n}{k} \frac{1}{2^n}$. 

```
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0 500 1000 1500 2000 2500 3000 3500 4000
n = 4096
```
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Consider flipping a fair coin \( n \) times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: \( k \) w.p. \( \binom{n}{k} \frac{1}{2^n} \).
Massive randomness.. Is not that random.

This is known as concentration of mass.
This is a very special case of the law of large numbers.
Informal statement of law of large numbers

For \( n \) large enough, the middle portion of the binomial distribution looks like (converges to) the normal/Gaussian distribution.
Massive randomness.. Is not that random.

**Intuitive conclusion**

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.
Massive randomness.. Is not that random.

Intuitive conclusion
Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.

Use of well known inequalities in analysis.
Analysis

- Random variable $Q = \#\text{comparisons}$ made by randomized QuickSort on an array of $n$ elements.

Suppose $\Pr[Q \geq 10n \log n] \leq c$. Also we know that $Q \leq n^2$. 

$$\mathbb{E}[Q] \leq (10n \log n)(1 - c) + n^2c$$

**Question:** How to find $c$, or in other words bound $\Pr[Q \geq 10n \log n]$?
Randomized **QuickSort**: A possible analysis

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How to find $c$, or in other words bound $\Pr[Q \geq 10n \log n]$?
Markov’s Inequality

Let $X$ be a non-negative random variable over a probability space $(\Omega, \Pr)$. For any $a > 0$,

$$\Pr[X \geq a] \leq \frac{E[X]}{a}$$
Markov’s Inequality

Let $X$ be a non-negative random variable over a probability space $(\Omega, \text{Pr})$. For any $a > 0$,

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$$

Proof:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \Pr[\omega] \geq \sum_{\omega \in \Omega, \ x(\omega) \geq a} X(\omega) \Pr[\omega] \geq a \sum_{\omega \in \Omega, \ x(\omega) \geq a} \Pr[\omega] = a \Pr[X \geq a]$$
Markov’s Inequality: Proof by Picture

\[ \text{Area} = \int_a^\infty x \cdot p_n(x) \, dx = E[X] \]

\[ \text{Area} = a \cdot p_n(x \geq a) \]
Example: Balls in a bin

- $n$ black and white balls in a bin.
- We wish to estimate the fraction of black balls. Let's say it is $p^*$. 
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- An approach: Draw $k$ balls with replacement. If $B$ are black then output $p = \frac{B}{k}$. 

Question: How large $k$ needs to be before our estimated value $p$ is close to $p^*$? 

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Example: Balls in a bin

A rough estimate through Markov’s inequality.

Lemma

For any \( k \geq 1 \) and \( p = \frac{B}{k} \), \( \Pr[p \geq 2p^*] \leq \frac{1}{2} \)
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**Lemma**

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**Proof.**

- For each $1 \leq i \leq k$ define random variable $X_i$, which is 1 if $i^{th}$ ball is black, otherwise 0.
- $E[X_i] = \Pr[X_i = 1] = p^*$.
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- $E[X_i] = \Pr[X_i = 1] = p^*$.
- $B = \sum_{i=1}^{k} X_i$, then $E[B] = \sum_{i=1}^{k} E[X_i] = kp^*$. 
Example: Balls in a bin

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- For each \( 1 \leq i \leq k \) define random variable \( X_i \), which is 1 if \( i^{th} \) ball is black, otherwise 0.

- \( E[X_i] = Pr[X_i = 1] = p^* \).

- \( B = \sum_{i=1}^{k} X_i \), then \( E[B] = \sum_{i=1}^{k} E[X_i] = kp^* \).

- Markov’s inequality gives, \( \Pr[p \geq 2p^*] = \)

\[
\Pr \left[ \frac{B}{k} \geq 2p^* \right] = \Pr[B \geq 2kp^*] = \Pr[B \geq 2 E[B]] \leq \frac{1}{2}
\]
Chebyshev’s Inequality: Variance

Variance

Given a random variable $X$ over probability space $(\Omega, \Pr)$, variance of $X$ is the measure of how much does it deviate from its mean value. Formally, $\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$.
Chebyshev’s Inequality: Variance

**Variance**

Given a random variable $X$ over probability space $(\Omega, \Pr)$, variance of $X$ is the measure of how much does it deviate from its mean value. Formally, $\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$.

**Intuitive Derivation**

Define $Y = (X - E[X])^2 = X^2 - 2X E[X] + E[X]^2$. 
Chebyshev’s Inequality: Variance

Variance

Given a random variable $X$ over probability space $(\Omega, \Pr)$, variance of $X$ is the measure of how much does it deviate from its mean value. Formally, 

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Intuitive Derivation

Define $Y = (X - \mathbb{E}[X])^2 = X^2 - 2X \mathbb{E}[X] + \mathbb{E}[X]^2$.

$$\text{Var}(X) = \mathbb{E}[Y]$$

$$= \mathbb{E}[X^2] - 2 \mathbb{E}[X] \mathbb{E}[X] + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
Independence
Random variables $X$ and $Y$ are called mutually independent if

$$\forall x, y \in \mathbb{R}, \quad \Pr[X = x \land Y = y] = \Pr[X = x] \Pr[Y = y]$$

Lemma

If $X$ and $Y$ are independent random variables then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$
Independence

Random variables $X$ and $Y$ are called mutually independent if

$$\forall x, y \in \mathbb{R}, \ Pr[X = x \land Y = y] = Pr[X = x] \ Pr[Y = y]$$

Lemma

If $X$ and $Y$ are independent random variables then

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Lemma

If $X$ and $Y$ are mutually independent, then $E[XY] = E[X] E[Y]$. 
Chebyshev’s Inequality

Given \( a \geq 0 \), \( \Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2} \)
Chebyshev’s Inequality

Given \( a \geq 0 \), \( \Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2} \)

Proof.

\( Y = (X - \mathbb{E}[X])^2 \) is a non-negative random variable. Apply Markov’s Inequality to \( Y \) for \( a^2 \).

\[
\Pr[Y \geq a^2] \leq \frac{\mathbb{E}[Y]}{a^2} \iff \Pr[(X - \mathbb{E}[X])^2 \geq a^2] \leq \frac{\text{Var}(X)}{a^2} \iff \Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}
\]
Chebyshev’s Inequality

Given $a \geq 0$, \( \Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2} \)

Proof.

\( Y = (X - E[X])^2 \) is a non-negative random variable. Apply Markov’s Inequality to \( Y \) for \( a^2 \).

\[
\Pr[Y \geq a^2] \leq \frac{E[Y]}{a^2} \iff \Pr[(X - E[X])^2 \geq a^2] \leq \frac{\text{Var}(X)}{a^2} \iff \Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}
\]

\[
\Pr[X \leq E[X] - a] \leq \frac{\text{Var}(X)}{a^2} \text{ AND } \Pr[X \geq E[X] + a] \leq \frac{\text{Var}(X)}{a^2}
\]
Lemma

For $0 < \epsilon < 1$, $k \geq 1$ and $p = \frac{B}{k}$, $\Pr[|p - p^*| \geq \epsilon] \leq \frac{1}{k\epsilon^2}$.

Proof.

Recall: $X_i$ is 1 if $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $\mathbb{E}[X_i] = p^*$, $\mathbb{E}[B] = kp^*$. $p = \frac{B}{k}$. 
Lemma

For $0 < \epsilon < 1$, $k \geq 1$ and $p = B/k$, $\Pr[|p - p^*| \geq \epsilon] \leq 1/k\epsilon^2$.

Proof.

Recall: $X_i$ is 1 if $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = B/k$. 

Example: Balls in a bin (contd)
Lemma

For $0 < \epsilon < 1$, $k \geq 1$ and $p = \frac{B}{k}$, $\Pr[|p - p^*| \geq \epsilon] \leq \frac{1}{k\epsilon^2}$.

Proof.

- Recall: $X_i$ is 1 if $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = \frac{B}{k}$.
- $\text{Var}(B) = \sum_i \text{Var}(X_i) = kp^*(1 - p^*)$ (Exercise)
Lemma

For $0 < \epsilon < 1$, $k \geq 1$ and $p = B/k$, $\Pr[|p - p^*| \geq \epsilon] \leq \frac{1}{k\epsilon^2}$.

Proof.

- Recall: $X_i$ is 1 if $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$.
- $Var(B) = \sum_i Var(X_i) = kp^*(1 - p^*)$ (Exercise)

\[
\Pr[|p - p^*| \geq \epsilon] = \Pr[|B/k - p^*| \geq \epsilon] = \Pr[|B - kp^*| \geq k\epsilon]
\]
**Lemma**

For $0 < \epsilon < 1$, $k \geq 1$ and $p = B/k$, $\Pr[|p - p^*| \geq \epsilon] \leq 1/k\epsilon^2$.

**Proof.**

- Recall: $X_i$ is 1 if $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = B/k$.
- $\text{Var}(B) = \sum_i \text{Var}(X_i) = kp^*(1 - p^*)$ (Exercise)

$$
\Pr[|p - p^*| \geq \epsilon] = \Pr[|B/k - p^*| \geq \epsilon]
= \Pr[|B - kp^*| \geq k\epsilon]
\leq \frac{\text{Var}(B)}{k^2\epsilon^2} = \frac{kp^*(1-p^*)}{k^2\epsilon^2}
< \frac{1}{k\epsilon^2}
$$

(Chebyshev)


Lemma

Let $X_1, \ldots, X_k$ be $k$ independent binary random variables such that, for each $i \in [1, k]$, $X_i$ equals 1 w.p. $p_i$, and 0 w.p. $(1 - p_i)$. 

Proof.

In notes!
Lemma

Let $X_1, \ldots, X_k$ be $k$ independent binary random variables such that, for each $i \in [1, k]$, $X_i$ equals 1 w.p. $p_i$, and 0 w.p. $(1 - p_i)$.

Let $X = \sum_{i=1}^{k} X_i$ and $\mu = \mathbb{E}[X] = \sum_i p_i$. 

For any $0 < \delta < 1$, it holds that:

$$\Pr\left[|X - \mu| \geq \delta \mu\right] \leq 2e^{-\delta^2 \mu/3}$$

and

$$\Pr\left[X \geq (1 + \delta) \mu\right] \leq e^{-\delta^2 \mu/3}$$

$$\Pr\left[X \leq (1 - \delta) \mu\right] \leq e^{-\delta^2 \mu/2}$$

Proof.

In notes!
Lemma

Let $X_1, \ldots, X_k$ be $k$ independent binary random variables such that, for each $i \in [1, k]$, $X_i$ equals 1 w.p. $p_i$, and 0 w.p. $(1 - p_i)$.

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Lemma

Let $X_1, \ldots, X_k$ be $k$ independent binary random variables such that, for each $i \in [1, k]$, $X_i$ equals $1$ w.p. $p_i$, and $0$ w.p. $(1 - p_i)$.

Let $X = \sum_{i=1}^{k} X_i$ and $\mu = E[X] = \sum_i p_i$.

For any $0 < \delta < 1$, it holds that:

$$\Pr[|X - \mu| \geq \delta \mu] \leq 2e^{-\delta^2 \mu \frac{3}{3}}$$

$$\Pr[X \geq (1 + \delta)\mu] \leq e^{-\delta^2 \mu \frac{3}{3}} \text{ and } \Pr[X \leq (1 - \delta)\mu] \leq e^{-\delta^2 \mu \frac{2}{2}}$$
Lemma

Let \( X_1, \ldots, X_k \) be \( k \) independent binary random variables such that, for each \( i \in [1, k] \), \( X_i \) equals 1 w.p. \( p_i \), and 0 w.p. \( 1 - p_i \).

Let \( X = \sum_{i=1}^{k} X_i \) and \( \mu = \mathbb{E}[X] = \sum_{i} p_i \).

For any \( 0 < \delta < 1 \), it holds that:

\[
\Pr[|X - \mu| \geq \delta \mu] \leq 2e^{-\frac{\delta^2 \mu}{3}}
\]

\[
\Pr[X \geq (1 + \delta)\mu] \leq e^{-\frac{\delta^2 \mu}{3}} \text{ and } \Pr[X \leq (1 - \delta)\mu] \leq e^{-\frac{\delta^2 \mu}{2}}
\]

Proof.

In notes!
Lemma

For any $0 < \epsilon < 1$, and $k \geq 1$, $\Pr[|p - p^*| \geq \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}}$.

Proof.

Recall: $X_i$ is $1$ is $i^{th}$ ball is black, else $0$.

$B = \sum_{i=1}^{k} X_i$. $\mathbb{E}[X_i] = p^*$, $\mathbb{E}[B] = kp^*$. $p = \frac{B}{k}$. 

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Lemma

For any $0 < \epsilon < 1$, and $k \geq 1$, $\Pr[|p - p^*| \geq \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}}$.

Proof.

Recall: $X_i$ is $1$ is $i^{th}$ ball is black, else $0$. $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = \frac{B}{k}$.

$$\Pr[|p - p^*| \geq \epsilon] = \Pr\left[\left|\frac{B}{k} - p^*\right| \geq \epsilon\right] = \Pr[|B - kp^*| \geq k\epsilon] = \Pr\left[|B - kp^*| \geq \left(\frac{\epsilon}{p^*}\right)kp^*\right]$$
Lemma

For any \(0 < \epsilon < 1\), and \(k \geq 1\), \(\Pr[|p - p^*| \geq \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}}\).

Proof.

Recall: \(X_i\) is 1 is \(i^{th}\) ball is black, else 0.

\(B = \sum_{i=1}^{k} X_i\). \(\mathbb{E}[X_i] = p^*\), \(\mathbb{E}[B] = kp^*\). \(p = \frac{B}{k}\).

\[
\Pr[|p - p^*| \geq \epsilon] = \Pr\left[\left|\frac{B}{k} - p^*\right| \geq \epsilon\right]
\]
\[
= \Pr[|B - kp^*| \geq k\epsilon]
\]
\[
= \Pr\left[|B - kp^*| \geq \left(\frac{\epsilon}{p^*}\right)kp^*\right]
\]

(Chernoff) \(\leq 2e^{-\frac{\epsilon^2}{3p^*^2}kp^*} = 2e^{-\frac{k\epsilon^2}{3p^*}}\)

\((p^* \leq 1) \leq 2e^{-\frac{k\epsilon^2}{3}}\)
Example Summary

The problem was to estimate the fraction of black balls $p^*$ in a bin filled with white and black balls. Our estimate was $p = \frac{B}{k}$ instead, where out of $k$ draws (with replacement) $B$ balls turns out black.

Markov’s Inequality

For any $k \geq 1$, $\Pr[p \geq 2p^*] \leq \frac{1}{2}$

Chebyshev’s Inequality

For any $0 < \epsilon < 1$, and $k \geq 1$, $\Pr[|p - p^*| \geq \epsilon] \leq \frac{1}{k\epsilon^2}$.

Chernoff Bound

For any $0 < \epsilon < 1$, and $k \geq 1$, $\Pr[|p - p^*| \geq \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}}$. 
Part IV

Randomized QuickSort (Contd.)
Randomized **QuickSort**: Recall

**Input:** Array $A$ of $n$ numbers. **Output:** Numbers in sorted order.

**Randomized QuickSort**

1. Pick a pivot element *uniformly at random* from $A$.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

**Note:** On every input randomized QuickSort takes $O(n \log n)$ time in expectation. On every input it may take $\Omega(n^2)$ time with some small probability.
Randomized **QuickSort**: Recall

**Input:** Array $A$ of $n$ numbers. **Output:** Numbers in sorted order.

**Randomized QuickSort**

1. Pick a pivot element *uniformly at random* from $A$.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
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**Note:** On *every* input randomized QuickSort takes $O(n \log n)$ time in expectation. On *every* input it may take $\Omega(n^2)$ time with some small probability.

**Question:** With what probability it takes $O(n \log n)$ time?
Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$. 
Informal Statement

Random variable $Q(A) =$ \# comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.

If $n = 100$ then this gives $\Pr[Q(A) \leq 32n \ln n] \geq 0.99999$. 
Randomized **QuickSort**: High Probability Analysis

**Informal Statement**
Random variable $Q(A) = \# \text{ comparisons done by the algorithm}$. We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - 1/n^3$.

**Outline of the proof**
- $k$: depth of the recursion. Then $Q(A) \leq kn$. 

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Randomized **QuickSort**: High Probability Analysis

**Informal Statement**
Random variable $Q(A) = \#$ comparisons done by the algorithm.
We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.

**Outline of the proof**
- $k$: depth of the recursion. Then $Q(A) \leq kn$.
- Prove that $k \leq 32 \ln n$ with high probability. Which will imply the result.
Randomized **QuickSort**: High Probability Analysis

### Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.

### Outline of the proof

- **$k$**: depth of the recursion. Then $Q(A) \leq kn$.
- Prove that $k \leq 32 \ln n$ with high probability. Which will imply the result.
  1. Focus on a single element. Prove that it “participates” in $> 32 \ln n$ levels with probability at most $\frac{1}{n^4}$.
  2. By union bound, any of the $n$ elements participates in $> 32 \ln n$ levels with probability at most $\frac{1}{n^3}$.
Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm. We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.

Outline of the proof

1. Focus on a single element. Prove that it “participates” in $> 32 \ln n$ levels with probability at most $\frac{1}{n^4}$.

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Randomized **QuickSort**: High Probability Analysis

**Informal Statement**
Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.

**Outline of the proof**

- **$k$**: depth of the recursion. Then $Q(A) \leq kn$.
- Prove that $k \leq 32 \ln n$ with high probability. Which will imply the result.

1. Focus on a single element. Prove that it “participates” in $> 32 \ln n$ levels with probability at most $\frac{1}{n^4}$.

2. By union bound, any of the $n$ elements participates in $> 32 \ln n$ levels with probability at most $\frac{1}{n^3}$.

3. Therefore, all elements participate in $\leq 32 \ln n$ w.p. $(1 - \frac{1}{n^3})$. 
Randomized QuickSort: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
If \( k \) levels of recursion then \( kn \) comparisons.

Fix an element \( s \in A \). We will track it at each level.

Let \( S_i \) be the partition containing \( s \) at \( i^{th} \) level.

\( S_1 = A \) and \( S_k = \{s\} \).
Randomized **QuickSort**: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
- Fix an element $s \in A$. We will track it at each level.
- Let $S_i$ be the partition containing $s$ at $i^{th}$ level.
- $S_1 = A$ and $S_k = \{s\}$.
- We call $s$ lucky in $i^{th}$ iteration, if balanced split:
  $$|S_{i+1}| \leq (3/4)|S_i| \text{ and } |S_i \setminus S_{i+1}| \leq (3/4)|S_i|.$$
Randomized QuickSort: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
- Fix an element $s \in A$. We will track it at each level.
- Let $S_i$ be the partition containing $s$ at $i^{th}$ level.
- $S_1 = A$ and $S_k = \{s\}$.
- We call $s$ lucky in $i^{th}$ iteration, if balanced split:
  $|S_{i+1}| \leq (3/4)|S_i|$ and $|S_i \setminus S_{i+1}| \leq (3/4)|S_i|$.
- If $\rho = \#$lucky rounds in first $k$ rounds, then
  $|S_k| \leq (3/4)^\rho n$.  
  
  For $|S_k| = 1$, $\rho = \log_4 \frac{4}{3} n \leq 4 \ln n$ suffices.
Randomized **QuickSort**: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
- Fix an element $s \in A$. We will track it at each level.
- Let $S_i$ be the partition containing $s$ at $i^{th}$ level.
- $S_1 = A$ and $S_k = \{ s \}$.
- We call $s$ lucky in $i^{th}$ iteration, if balanced split: $|S_{i+1}| \leq (3/4)|S_i|$ and $|S_i \setminus S_{i+1}| \leq (3/4)|S_i|$.
- If $\rho = \#\text{lucky rounds in first } k\text{ rounds}$, then $|S_k| \leq (3/4)^\rho n$.
- For $|S_k| = 1$, $\rho = \log_{4/3} n \leq 4 \ln n$ suffices.
How many rounds before $4 \ln n$ lucky rounds?

$s$ lucky in round $i$ if $|S_{i+1}| \leq (3/4)|S_i|$ and $|S_i \setminus S_{i+1}| \leq (3/4)|S_i|$
How many rounds before $4 \ln n$ lucky rounds?

$s$ lucky in round $i$ if $|S_{i+1}| \leq (3/4)|S_i|$ and $|S_i \setminus S_{i+1}| \leq (3/4)|S_i|$

- $X_i = 1$ if $s$ is lucky in $i^{th}$ round.
How many rounds before $4 \ln n$ lucky rounds?

$s$ lucky in round $i$ if $|S_{i+1}| \leq (3/4)|S_i|$ and $|S_i \setminus S_{i+1}| \leq (3/4)|S_i|$

- $X_i = 1$ if $s$ is lucky in $i^{th}$ round.
- Observation: $X_1, \ldots, X_k$ are independent variables.
- $\Pr[X_i = 1] = \frac{1}{2}$ Why?

Clearly, $\rho = \sum_{i=1}^{k} X_i$. Let $\mu = E[\rho] = k/2$.

Set $k = 32 \ln n$ and $\delta = 3/4$.

$(1 - \delta)^2 = 1/4$.

Probability of NOT getting $4 \ln n$ lucky rounds out of $32 \ln n$ rounds

$\Pr[\rho \leq 4 \ln n] = \Pr[\rho \leq k/8] \leq \Pr[\rho \leq (1 - \delta) \mu]$

(Chernoff) $\leq 2e^{-\delta^2 \mu} = 2e^{-4 \log n} = 2e^{-4 \ln n} = 1/n^4$. 
How many rounds before \(4 \ln n\) lucky rounds?

\(s\) lucky in round \(i\) if \(|S_{i+1}| \leq (3/4)|S_i|\) and \(|S_i \setminus S_{i+1}| \leq (3/4)|S_i|\)

- \(X_i = 1\) if \(s\) is lucky in \(i^{th}\) round.
- **Observation:** \(X_1, \ldots, X_k\) are independent variables.
- \(\Pr[X_i = 1] = \frac{1}{2}\) Why?
- Clearly, \(\rho = \sum_{i=1}^{k} X_i\). Let \(\mu = \mathbb{E}[\rho] = \frac{k}{2}\).
How many rounds before $4 \ln n$ lucky rounds?

$s$ lucky in round $i$ if $|S_{i+1}| \leq (3/4)|S_i|$ and $|S_i \setminus S_{i+1}| \leq (3/4)|S_i|$

- $X_i = 1$ if $s$ is lucky in $i^{th}$ round.
- **Observation:** $X_1, \ldots, X_k$ are independent variables.
- $\Pr[X_i = 1] = \frac{1}{2}$  Why?

Clearly, $\rho = \sum_{i=1}^{k} X_i$. Let $\mu = \mathbb{E}[\rho] = \frac{k}{2}$.

Set $k = 32 \ln n$ and $\delta = \frac{3}{4}$. $(1 - \delta) = \frac{1}{4}$. 

How many rounds before $4 \ln n$ lucky rounds?

$s$ lucky in round $i$ if $|S_{i+1}| \leq (3/4)|S_i|$ and $|S_i \setminus S_{i+1}| \leq (3/4)|S_i|$

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- $\Pr[X_i = 1] = \frac{1}{2}$  Why?
- Clearly, $\rho = \sum_{i=1}^{k} X_i$. Let $\mu = \mathbb{E}[\rho] = \frac{k}{2}$.
- Set $k = 32 \ln n$ and $\delta = \frac{3}{4}$. $(1 - \delta) = \frac{1}{4}$.

Probability of NOT getting $4 \ln n$ lucky rounds out of $32 \ln n$ rounds
How many rounds before \(4 \ln n\) lucky rounds?

\(s\) lucky in round \(i\) if \(|S_{i+1}| \leq (3/4)|S_i|\) and \(|S_i \setminus S_{i+1}| \leq (3/4)|S_i|\)

- \(X_i = 1\) if \(s\) is lucky in \(i^{th}\) round.
- **Observation:** \(X_1, \ldots, X_k\) are independent variables.
- \(\Pr[X_i = 1] = \frac{1}{2}\) Why?

Clearly, \(\rho = \sum_{i=1}^{k} X_i\). Let \(\mu = E[\rho] = \frac{k}{2}\).

Set \(k = 32 \ln n\) and \(\delta = \frac{3}{4}\). \((1 - \delta) = \frac{1}{4}\).

Probability of NOT getting \(4 \ln n\) lucky rounds out of \(32 \ln n\) rounds

\[
\Pr[\rho \leq 4 \ln n] = \Pr[\rho \leq \frac{k}{8}] = \Pr[\rho \leq (1 - \delta)\mu]
\]
How many rounds before $4 \ln n$ lucky rounds?

$s$ lucky in round $i$ if $|S_{i+1}| \leq (3/4)|S_i|$ and $|S_i \setminus S_{i+1}| \leq (3/4)|S_i|$

- $X_i = 1$ if $s$ is lucky in $i^{th}$ round.
- **Observation:** $X_1, \ldots, X_k$ are independent variables.
- $\Pr[X_i = 1] = \frac{1}{2}$ Why?
- Clearly, $\rho = \sum_{i=1}^{k} X_i$. Let $\mu = \mathbb{E}[\rho] = \frac{k}{2}$.
- Set $k = 32 \ln n$ and $\delta = \frac{3}{4}$. $(1 - \delta) = \frac{1}{4}$.

Probability of NOT getting $4 \ln n$ lucky rounds out of $32 \ln n$ rounds

$$\Pr[\rho \leq 4 \ln n] = \Pr[\rho \leq \frac{k}{8}] = \Pr[\rho \leq (1 - \delta)\mu] \leq 2e^{-\frac{\delta^2\mu}{2}} = 2e^{-\frac{9k}{64}} = 2e^{-4.5 \ln n} \leq \frac{1}{n^4}$$
Randomized **QuickSort** w.h.p. Analysis

- n input elements. Probability that there is some un-lucky element is at most \( \frac{1}{n^4} \times n = \frac{1}{n^3} \).
Randomized **QuickSort** w.h.p. Analysis

- n input elements. Probability that there is some un-lucky element is at most \( \frac{1}{n^4} \times n = \frac{1}{n^3} \).
- \( \Pr[\text{depth of recursion in QuickSort} > 32 \ln n] \leq \frac{1}{n^3} \).
Randomized QuickSort w.h.p. Analysis

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*With high probability (i.e., $1 - \frac{1}{n^3}$) the depth of the recursion of QuickSort is $\leq 32 \ln n$. Due to $n$ comparisons in each level, with high probability, the running time of QuickSort is $O(n \ln n)$.***
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Q: How to increase the probability?