CS 473: Algorithms

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Spring 2021

Polynomials, Convolutions and FFT

Lecture 2

Jan 26/28, 2021

Most slides are courtesy Prof. Chekuri

Outline

Discrete Fourier Transfor (DFT) and Fast Fourier Transform (FFT) have many applications and are connected to important mathematics.

"One of top 10 Algorithms of 20th Century" according to IEEE.
Gilbert Strang: "The most important numerical algorithm of our lifetime".

Our goal:

- Multiplication of two degree n polynomials in $O(n \log n)$ time. Surprising and non-obvious.
- Algorithmic ideas
  - change in representation
  - mathematical properties of polynomials
  - divide and conquer

Part I

Polynomials, Convolutions and FFT

Polynomials

Definition

A polynomial is a function of one variable built from additions, subtractions and multiplications (but no divisions).

$$p(x) = \sum_{j=0}^{n-1} a_j x^j$$

The numbers $a_0, a_1, \ldots, a_n$ are the coefficients of the polynomial. The degree is the highest power of x with a non-zero coefficient.

Example

$$p(x) = 3 - 4x + 5x^3$$

$a_0 = 3, a_1 = -4, a_2 = 0, a_3 = 5$ and $deg(p) = 3$

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Coefficient Representation

Polynomials represented by vector $a = (a_0, a_1, \ldots a_{n-1})$ of coefficients.

Operations on Polynomials

- **Evaluate** Given a polynomial p and a value α, compute $p(\alpha)$
- **Add** Given (representations of) polynomials p, q, compute (reprsentation of) polynomial $p + q$
- **Multiply** Given (representation of) polynomials p, q, compute (representation of) polynomial $p \cdot q$.
- **Roots** Given p find all *roots* of p.

Evaluation

Compute value of polynomial $a = (a_0, a_1, \ldots a_{n-1})$ at α

```
power = 1
value = 0
for j = 0 to n − 1
    // invariant:  power = α^j
    value = value + a_j · power
    power = power · α
end for
return value
```

How many additions?

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How many additions? n

How many multiplications? $2n$

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How many additions? n

How many multiplications? $2n$

Horner's rule can be used to cut the multiplications in half

$$a(x) = a_0 + x(a_1 + x(a_2 + x(\cdots + xa_{n-1})\cdots))$$

Evaluation: Numerical Issues

Question

How long does evaluation really take? $O(n)$ time?

Bits to represent α^n is $n \log \alpha$ while bits to represent α is only $\log \alpha$. Thus, need to pay attention to size of numbers and multiplication complexity.

Ignore this issue for now. Can get around it for applications of interest where one typically wants to compute $p(\alpha) \mod m$ for some number m.

Addition

Compute the sum of polynomials
$a = (a_0, a_1, \ldots a_{n-1})$ and $b = (b_0, b_1, \ldots b_{n-1})$

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$a = (a_0, a_1, \ldots a_{n-1})$ and $b = (b_0, b_1, \ldots b_{n-1})$
$a + b = (a_0 + b_0, a_1 + b_1, \ldots a_{n-1} + b_{n-1})$. Takes $O(n)$ time.

Multiplication

Compute the product of polynomials
$a = (a_0, a_1, \ldots a_n)$ and $b = (b_0, b_1, \ldots b_m)$
Recall $a \cdot b = (c_0, c_1, \ldots c_{n+m})$ where

$$c_k = \sum_{i,j:\, i+j=k} a_i \cdot b_j$$

Takes $\Theta(nm)$ time; $\Theta(n^2)$ when $n = m$.

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We will obtain a better algorithm!

Better/Efficient/Easy (today's lecture): preferably $O(n + m)$, but $O(n \log n)$ is also okay.

Convolutions

Definition

The convolution of vectors $a = (a_0, a_1, \ldots a_n)$ and $b = (b_0, b_1, \ldots b_m)$ is the vector $c = (c_0, c_1, \ldots c_{n+m})$ where

$$c_k = \sum_{i,j:\, i+j=k} a_i \cdot b_j$$

Convolution of vectors a and b is denoted by $a * b$. In other words, the convolution is the coefficients of the product of the two polynomials.

Revisiting Polynomial Representations

Representation

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Question

Are there other useful ways to represent polynomials?

Representing Polynomials by Roots

Root of a polynomial $p(x)$: r such that $p(r) = 0$. If $r_1, r_2, \ldots, r_{n-1}$ are roots then $p(x) = a_{n-1}(x - r_1)(x - r_2)\ldots(x - r_{n-1})$.

Valid representation because of:

Theorem (Fundamental Theorem of Algebra)

Every polynomial $p(x)$ of degree d has exactly d roots $r_1, r_2, \ldots, r_d$ where the roots can be complex numbers and can be repeated.

Representing Polynomials by Roots

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- Evaluating p at a given x is easy. Why?

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Polynomials represented by vector scale factor a_{n-1} and roots $r_1, r_2, \ldots, r_{n-1}$.

- Evaluating p at a given x is easy. Why?
- Multiplication: given p, q with roots $r_1, \ldots, r_{n-1}$ and $s_1, \ldots, s_{m-1}$ the product $p \cdot q$ has roots $r_1, \ldots, r_{n-1}, s_1, \ldots, s_{m-1}$. Easy! $O(n+m)$ time.

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- Addition: requires $\Omega(nm)$ time?

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- Evaluating p at a given x is easy. Why?
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- Addition: requires $\Omega(nm)$ time?
- Given coefficient representation, how do we go to root representation? No finite algorithm because of potential for irrational roots.

Representing Polynomials by Samples

Let p be a polynomial of degree $n-1$.
Pick n *distinct samples* $x_0, x_1, x_2, \ldots, x_{n-1}$
Let $y_0 = p(x_0), y_1 = p(x_1), \ldots, y_{n-1} = p(x_{n-1})$.

Representation

Polynomials represented by $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$.

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Is the above a valid representation?

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Representation

Polynomials represented by $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})$.

Is the above a valid representation? Why do we use $2n$ numbers instead of n numbers for coefficient and root representation?

Sample Representation

Theorem

Given a list $\{(x_0, y_0), (x_1, y_1), \ldots (x_{n-1}, y_{n-1})\}$ *there is exactly one polynomial* p *of degree* $n-1$ *such that* $p(x_j) = y_j$ *for* $j = 0, 1, \ldots, n-1$.

Sample Representation

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So representation is valid.

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So representation is valid.

Can use same $x_0, x_1, \ldots, x_{n-1}$ for all polynomials of degree $n-1$.

No need to store them explicitly and hence need only n numbers $y_0, y_1, \ldots, y_{n-1}$.

(

Lagrange Interpolation

Given $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$ the following polynomial p satisfies the property that $p(x_j) = y_j$ for $j = 0, 1, 2, \ldots, n-1$.

$$p(x) = \sum_{j=0}^{n-1} \left(\frac{y_j}{\prod_{k \neq j}(x_j - x_k)} \prod_{k \neq j}(x - x_k) \right)$$

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For $n = 3$, $p(x) =$

$$y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

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Easy to verify that $p(x_j) = y_j$! Thus there exists one polynomial of degree $n - 1$ that interpolates the values $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$.

Lagrange Interpolation

Given $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$ there is a polynomial $p(x)$ such that $p(x_i) = y_i$ for $0 \leq i < n$. Can there be two distinct polynomials?

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No! Use Fundamental Theorem of Algebra to prove it — exercise.

Addition and Multiplication with Sample Representation

- Let $a = \{(x_0, y_0), (x_1, y_1), \ldots (x_{n-1}, y_{n-1})\}$ and $b = \{(x_0, y'_0), (x_1, y'_1), \ldots (x_{n-1}, y'_{n-1})\}$ be two polynomials of degree $n-1$ in sample representation.

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- $a + b$ can be represented by

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- $a + b$ can be represented by $\{(x_0, (y_0 + y'_0)), (x_1, (y_1 + y'_1)), \ldots (x_{n-1}, (y_{n-1} + y'_{n-1}))\}$
  - Thus, can be computed in $O(n)$ time

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  - Thus, can be computed in $O(n)$ time
- $a \cdot b$ can be evaluated at n samples

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  - Thus, can be computed in $O(n)$ time
- $a \cdot b$ can be evaluated at n samples $\{(x_0, (y_0 \cdot y_0')), (x_1, (y_1 \cdot y_1')), \ldots (x_{n-1}, (y_{n-1} \cdot y_{n-1}'))\}$
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  - Can be computed in $O(n)$ time.

But what if p, q are given in coefficient form? Convolution requires p, q to be in coefficient form.

Recall

Goal: given polynomials $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$ in coefficient representation, compute $a \cdot b$ in coefficient form **(convolution)**.

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Sample representation:

- Fix $x_0, \ldots, x_{n-1}$.
- $a' = (x_0, a(x_0)), \ldots, (x_{n-1}, a(x_{n-1}))$, similarly b' from b.
  - **Theorem.** Unique degree $(n-1)$ polynomial corresponding to any given n samples.

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- $a' \cdot b'$ requires $O(n)$ multiplications.

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  - **Theorem.** Unique degree $(n-1)$ polynomial corresponding to any given n samples. a' is a *valid* representation of a.
- $a' \cdot b'$ requires $O(n)$ multiplications.

Plan. Convert to sample representation. Multiply. Convert back to coefficient representation.

Coefficient representation to Sample representation

Given a polynomial a as $(a_0, a_1, \ldots, a_{n-1})$ can we obtain a sample representation $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$ quickly? Also can we *invert* the representation quickly?

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- Suppose we choose $x_0, x_1, \ldots, x_{n-1}$ arbitrarily.
- Take $O(n)$ time to evaluate $y_j = a(x_j)$ given $(a_0, \ldots, a_{n-1})$.
- Total time is $\Omega(n^2)$
- Inversion via Lagrange interpolation also $\Omega(n^2)$

Key Idea

Can choose $x_0, x_1, \ldots, x_{n-1}$ carefully!

Total time to evaluate $a(x_0), a(x_1), \ldots, a(x_{n-1})$ should be better than evaluating each separately.

Key Idea

Can choose $x_0, x_1, \ldots, x_{n-1}$ carefully!

Total time to evaluate $a(x_0), a(x_1), \ldots, a(x_{n-1})$ should be better than evaluating each separately.

How do we choose $x_0, x_1, \ldots, x_{n-1}$ to save work?

A Simple Start

$$a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{n-1} x^{n-1}$$

Assume n is a power of 2 for rest of the discussion.

Observation: $(-x)^{2j} = x^{2j}$. Can we exploit this?

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Example

$3+4x+6x^2+2x^3+x^4+10x^5 = (3+6x^2+x^4)+x(4+2x^2+10x^4)$

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$a(c) = (3 + 6c^2 + c^4) + c(4 + 2c^2 + 10c^4)$

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$a(c) = (3 + 6c^2 + c^4) + c(4 + 2c^2 + 10c^4)$
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Example

$$3+4x+6x^2+2x^3+x^4+10x^5 = (3+6x^2+x^4)+x(4+2x^2+10x^4)$$

$$a(c) = (3 + 6c^2 + c^4) + c(4 + 2c^2 + 10c^4)$$
$$a(-c) = (3 + 6c^2 + c^4) - c(4 + 2c^2 + 10c^4)$$

Odd and Even Decomposition

- Let $a = (a_0, a_1, \ldots a_{n-1})$ be a polynomial.
- Let $a_{\mathrm{odd}} = (a_1, a_3, a_5, \ldots)$ be the $(n/2 - 1)$ degree polynomial defined by the odd coefficients; so

$$a_{\mathrm{odd}}(x) = a_1 + a_3 x + a_5 x^2 + \cdots$$

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- Similarly, let $a_{\text{even}}(x) = a_0 + a_2 x + \ldots$ be the $(n/2 - 1)$ degree polynomial defined by the even coefficients.

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$$a_{\rm odd}(x) = a_1 + a_3 x + a_5 x^2 + \cdots$$

- Similarly, let $a_{\rm even}(x) = a_0 + a_2 x + \ldots$ be the $(n/2 - 1)$ degree polynomial defined by the even coefficients.
- Observe

$$a(x) = a_{\rm even}(x^2) + x a_{\rm odd}(x^2)$$

- Thus, evaluating a at x can be reduced to evaluating lower degree polynomials plus constantly many arithmetic operations.

Exploiting Odd-Even Decomposition

$$a(x) = a_{\text{even}}(x^2) + x a_{\text{odd}}(x^2)$$

- Choose n samples
  $x_0, x_1, x_2, \ldots, x_{n/2-1}, -x_0, -x_1, \ldots, -x_{n/2-1}$
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  Total of $O(n)$ work!
- Suppose we can make this work recursively. Then

$$T(n) = 2T(n/2) + O(n) \text{ which implies } T(n) = O(n \log n)$$

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Given a set X of numbers square(X) (for square of X) is the set $\{x^2 \mid x \in X\}$.

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A set X of n numbers (for n a power of 2) is *recursively collapsible* if $n = 1$ or if X is collapsible and $\text{square}(X)$ is recursively collapsible.

Divide and Conquer assuming collapsible set

Given a *recursively collapsible* set X of size n, compute sample representation of polynomial a of degree $(n-1)$ as follows:

SampleRepresentation(a, X, n)

```
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Exercise: show that algorithm runs in $O(n \log n)$ time

Are there collapsible sets?

- n samples $x_0, x_1, x_2, \ldots, x_{n/2-1}, -x_0, -x_1, \ldots, -x_{n/2-1}$
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- If $z_0 = x_0^2$ and $-z_0 = x_{n/4}^2$ then $x_0 = \sqrt{-1}x_{n/4}$ That is $x_0 = ix_{n/4}$ where i is the imaginary number.

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- Can continue recursion but need to go to *complex* numbers.

Complex Numbers

Notation

For the rest of lecture, i stands for $\sqrt{-1}$

Definition

Complex numbers are points lying in the complex plane represented as

Cartesian $a + ib = \sqrt{a^2 + b^2} e^{(\arctan(b/a))i}$

Polar $re^{\theta i} = r(\cos\theta + i\sin\theta)$

Thus, $e^{\pi i} = -1$ and $e^{2\pi i} = 1$.

Power Series for Functions (Recall)

What is e^z when z is a real number? When z is a complex number?

$$e^z = 1 + z/1! + z^2/2! + \ldots + z^j/j! + \ldots$$

Therefore

$$\begin{aligned} e^{i\theta} &= 1 + i\theta/1! + (i\theta)^2/2! + (i\theta)^3/3! + \ldots \\ &= (1 - \theta^2/2! + \theta^4/4! - \ldots +) + i(\theta - \theta^3/3! + \ldots +) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Complex Roots of Unity

What are the roots of the polynomial $x^k - 1$? $(e^{2\pi i} = 1)$

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- Let $\omega_k = e^{2\pi i/k}$. The roots are $1 = \omega_k^0, \omega_k^2, \ldots, \omega_k^{k-1}$ where $\omega_k^j = e^{2\pi j i/k}$.

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Proposition

Let ω_k be $e^{2\pi i/k}$. The equation $x^k = 1$ has k distinct complex roots given by $\omega_k^j = e^{(2\pi j)i/k}$ for $j = 0, 1, \ldots k - 1$

Proof.

$(\omega_k^j)^k = (e^{2\pi j i/k})^k = e^{2\pi j i} = (e^{2\pi i})^j = (1)^j = 1$ $\square$

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Observation 1: $\omega_k^j = \omega_k^{j \bmod k}$

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- $X_1 = \{1\}$, $X_2 = \{1, -1\}$
- $X_4 = \{1, -1, i, -i\}$
- $X_8 = \{1, -1, i, -i, \frac{1}{\sqrt{2}}(\pm 1 \pm i)\}$

Discrete Fourier Transform

Definition

Given vector $a = (a_0, a_1, \ldots, a_{n-1})$ the *Discrete Fourier Transform* (DFT) of a is the vector $a' = (a'_0, a'_1, \ldots, a'_{n-1})$ where $a'_j = a(\omega_n^j)$ for $0 \leq j < n$.

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We have shown that a' can be computed from a in $O(n \log n)$ time. This divide and conquer *algorithm* is called the *Fast Fourier Transform* (FFT).

Back to Convolutions and Polynomial Multiplication

Convolutions (products)

Compute convolution $c = (c_0, c_1, \ldots, c_{2n-2})$ of $a = (a_0, a_1, \ldots a_{n-1})$ and $b = (b_0, b_1, \ldots b_{n-1})$

1. Evaluate a and b at some n sample points.
2. Compute sample representation of product. That is $c' = (a'_0 b'_0, a'_1 b'_1, \ldots, a'_{n-1} b'_{n-1})$.
3. Compute coefficients of unique polynomial associated with sample representation of product. That is compute c from c'.

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Can we really compute c from c'? We only have n sample points and c' has $2n-1$ coefficients!

Convolutions and Polynomial Multiplication

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- Step 2 takes $O(n \log n)$ using divide and conquer algorithm
- Step 3 takes $O(n)$ time
- Step 4?

Part II

Inverse Fourier Transform

Inverse Fourier Transform

Input Given the evaluation of a $n-1$-degree polynomial a on the nth roots of unity specified by vector a'

Goal Compute the coefficients of a

We saw that a' can be computed from a in $O(n \log n)$ time. Can we compute a from a' in $O(n \log n)$ time?

A Matrix Point of View

$$a(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$

$a'_0 = a(x_0), a'_1 = a(x_1), \ldots, a'_{n-1} = a(x_{n-1})$.

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_j & x_j^2 & \ldots & x_j^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \ldots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a'_0 \\ a'_1 \\ \vdots \\ a'_j \\ \vdots \\ a'_{n-1} \end{bmatrix}$$

A Matrix Point of View

$$a(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$$

Denote $\omega = \omega_n^1 = e^{2\pi/n}$. Let $x_j = \omega^j$

$a'_0 = a(1), a'_1 = a(\omega), \ldots, a'_{n-1} = a(\omega^{n-1})$.

$$\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^j & \omega^{2j} & \ldots & \omega^{j(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a'_0 \\ a'_1 \\ \vdots \\ a'_j \\ \vdots \\ a'_{n-1} \end{bmatrix}$$

Inverting the Matrix

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^j & \omega^{2j} & \dots & \omega^{j(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} a'_0 \\ a'_1 \\ \vdots \\ a'_j \\ \vdots \\ a'_{n-1} \end{bmatrix}$$

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Replace ω by ω^{-1} which is also a root of unity!
Since $\omega^j = \omega^{j \bmod n}$, we get $\omega^{-j} = e^{-j2\pi/n} = \omega^{(n-j)2\pi/n}$.

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Inverse matrix is simply a permutation of the original matrix modulo scale factor $1/n$.

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Check $VV^{-1} = I$ where I is the $n \times n$ identity matrix.

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$$(1, \omega^j, \omega^{2j}, \ldots, \omega^{j(n-1)}) \cdot (1, \omega^{-k}, \omega^{-2k}, \ldots, \omega^{-k(n-1)}) = \sum_{s=0}^{n-1} \omega^{s(j-k)}$$

Note that ω^{j-k} is a n'th root of unity. If $j = k$ then sum is n, otherwise by previous observation sum is $\mathbf{0}$.

Why does it work?

Check $VV^{-1} = I$ where I is the $n \times n$ identity matrix.

Observation: $\sum_{s=0}^{n-1} (\omega^j)^s = (1 + \omega^j + \omega^{2j} + \ldots + \omega^{(n-1)j}) = 0$, $j \neq 0$

- ω^j is root of $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + 1)$
- Thus, ω^j is root of $(x^{n-1} + x^{n-2} + \ldots + 1)$

$$(1, \omega^j, \omega^{2j}, \ldots, \omega^{j(n-1)}) \cdot (1, \omega^{-k}, \omega^{-2k}, \ldots, \omega^{-k(n-1)}) = \sum_{s=0}^{n-1} \omega^{s(j-k)}$$

Note that ω^{j-k} is a n'th root of unity. If $j = k$ then sum is n, otherwise by previous observation sum is 0.

Rows of matrix V (and hence also those of V^{-1}) are *orthogonal*. Thus $a' = Va$ can be thought of transforming the vector a into a new Fourier basis with basis vectors corresponding to rows of V.

Inverse Fourier Transform

Input: Given the evaluation of a $n-1$-degree polynomial a on the nth roots of unity specified by vector a'

Goal: Compute the coefficients of a

We saw that a' can be computed from a in $O(n \log n)$ time. Can we compute a from a' in $O(n \log n)$ time?

Yes! $a = V^{-1}a'$ which is simply a permuted and scaled version of DFT. Hence can be computed in $O(n \log n)$ time.

Convolutions Once More

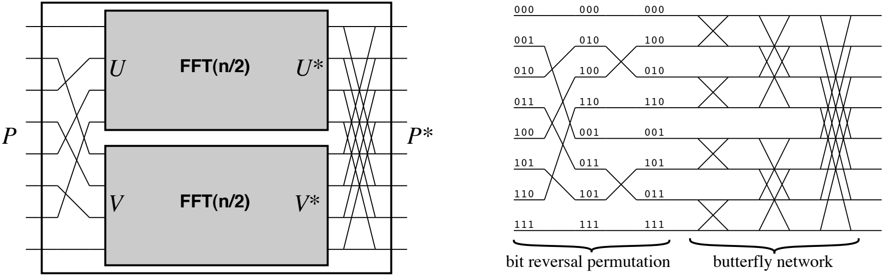
Convolutions

Compute convolution of $a = (a_0, a_1, \ldots a_{n-1})$ and $b = (b_0, b_1, \ldots b_{n-1})$

1. Compute values of a and b at the $2n$th roots of unity
2. Compute sample representation c' of product $c = a \cdot b$
3. Compute c from c' using inverse Fourier transform.

- Step 1 takes $O(n \log n)$ using two FFTs
- Step 2 takes $O(n)$ time
- Step 3 takes $O(n \log n)$ using one FFT

FFT Circuit



The recursive structure of the FFT algorithm.

Numerical Issues

- As noted earlier evaluating a polynomial p at a point x makes numbers big
- Are we cheating when we say $O(n \log n)$ algorithm for convolution?
- Can get around numerical issues — work in finite fields and avoid numbers growing too big.
- Outside the scope of lecture
- We will assume for reductions that convolution can be done in $O(n \log n)$ time.

Numerical Issues: Puzzle

Part III

Application to String Matching

Basic string matching problem:

- Input: Given a pattern string P on length m and a text string T of length n over a fixed alphabet Σ
- Goal: Does P occur as a substring of T? Find all "matches" of P in T.

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Several generalizations. Matching with don't cares.

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Example: $P = a**$, $T = aardvark$

Matches?

Shifted products via Convolution

Given two arrays A and B with say with $A[0..m-1]$ and $B[0..n-1]$ with $m \leq n$

Input Two arrays: $A[0..(m-1)]$ and $B[0..(n-1)]$.

Goal Compute all shifted products in array $C[0..(n-m-1)]$ where $C[i] = \sum_{j=0}^{m-1} A[j]B[i+j]$.

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Lemma

Reverse *of* C *is the convolution of the vectors* A *and* reverse *of* B.

Proof.

Exercise. □

Reduction of pattern matching to shifted products

Assume first that $\Sigma = \{0, 1\}$

Goal:

- Convert $P = a_0 a_1 \ldots a_{m-1}$ to binary array A of size m.
- Convert $T = b_0 b_1 \ldots b_{n-1}$ to binary array B of size n.
- So that we can use shifted product C of A and B to count "mismatches".

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- Finding Type 1 mismatches:
  - $B[j] = T[j]$
  - If $P[j] = 0$ set $A[j] = 1$, if $P[j] = 1$ or $*$ set $A[j] = 0$.

Reduction of pattern matching to shifted products

- **Type 2** mismatches: $C[i]$ counts # j's where $P[j] = 1$ and $T[i+j] = 0$, when P is aligned with T at $T[i]$.

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There is a match at position i of T iff both types of mismatches are 0.

Running time analysis

- Reducing to shift product is $O(n)$.
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- Need to compute two convolutions with polynomials of size n and m. Total run time is $O(n \log n)$ (here we assume $m \leq n$).
- Can reduce to $O(n \log m)$ as follows. Break text T into $O(n/m)$ *overlapping* substrings of length $2m$ each and compute matches of P with these substrings. Total time is $O(n \log m)$.

Exercise: work out the details of this improvement.

General Alphabet

If Σ is not binary replace each character $\alpha \in \Sigma$ by its binary representation. Need $s = \lceil \log |\Sigma| \rceil$ bits. Running time increases to $O(n \log m \log s)$.

Can remove dependence on s and obtain $O(n \log m)$ time where $m = |P|$ using more advanced ideas and/or randomization.

Trivia

FFT algorithm is used billions of times everyday: image/sound processing – jpeg, mp3, MRI scans, etc.

Even your brain is running FFT!

A fun video on FFT applications:
https://www.youtube.com/watch?v=aqa6vyGSdos