LECTURE 7 (September 16th)

More Probability & Randomized Algorithms

RECAP **Equality Testing** Given two binary vectors $u, v \in \{0,1\}^n$
Decide if they are equal or not

Only operation that is allowed : DOTPRODUCT (a, b) → Time $B(n)$

take dot product (mod 2) of any two binary vectors $a, b \in \{0,1\}^n$
i.e. output is

$$\underbrace{\langle a,b\rangle}_{=\sum_{i=1}^{n} a_i b_i} \bmod 2 = \sum_{i=1}^{n} a_i b_i \pmod 2 = \begin{cases} 1 & \text{if } \langle a,b\rangle \text{ is odd} \\ 0 & \text{o/w} \end{cases}$$

Deterministically Let $e_i = [0 0 \cdots \overset{i^{th}\text{ coordinate}}{\overset{\downarrow}{1}} 0 \cdots 0]$ be the i^{th}-standard basis vector
Invoking DOTPRODUCT (u, e_i) for $i = 1$ to n, tells us what u or v is
Time $= O(n \cdot B(n))$

Algorithm

- Pick a random vector $r \in \{0,1\}^n$
- If $\langle u,r\rangle = \langle v,r\rangle \bmod 2$, then output EQUAL
- Else output NOT EQUAL

Theorem $\mathbb{P}[\text{Algorithm errs}] \le \frac{1}{2}$ and is running time is $\underbrace{O(n + B(n))}_{\text{obvious}}$

Proof Algorithm only errs if $u \neq v$

suppose u and v differ on the last bit : $u_n \neq v_n$

Then, $\langle u,r\rangle = \underbrace{\sum_{i=1}^{n-1} u_i r_i}_{\alpha} + u_n r_n$

$$\langle v,r\rangle = \underbrace{\sum_{i=1}^{n-1} v_i r_i}_{\beta} + v_n r_n$$

Now, there are two cases

$\boxed{1}$ $\underline{\alpha \neq \beta \bmod 2}$ w.p. $\frac{1}{2}$ $r_n = 0$, so $\langle u,r\rangle \neq \langle v,r\rangle$

$\boxed{2}$ $\underline{\alpha = \beta \bmod 2}$ w.p. $\frac{1}{2}$ $r_n = 1$, so $\langle u,r\rangle \neq \langle v,r\rangle$

Thus, $\mathbb{P}[\text{Algorithm errs}] \le \frac{1}{2}$ ← This is not very small
Can we make it $\le \delta$?

Repetition/Amplification Trick Run the algorithm $t = \lceil \log \frac{1}{\delta} \rceil$ times independently
If any execution says NOT EQUAL ⇒ output NOT EQUAL
o/w ⇒ output EQUAL

Again, algorithm only errs if $u \neq v$,

$$\mathbb{P}[\text{Algorithm errs}] = \mathbb{P}[\text{all } i \text{ iteration return EQUAL}]$$
$$= \prod_{i=1}^{t} \frac{1}{2} = 2^{-t} = 2^{-\lceil \log 1/\delta \rceil} \leq \delta$$

Runtime is now $O((n + B(n)) \log \frac{1}{\delta})$

Testing Matrix Product Given Boolean matrices $B, C, D \in \{0,1\}^{n \times n}$
decide if $BC = D \pmod 2$

Matrix Multiplication takes $O(n^{2.3\cdots})$ time.

Randomness allows us to do it in roughly $O(n^2)$ time.

Algorithm Take a random Boolean vector $r \in \{0,1\}^n$

- Compute (mod 2) $Dr = y$
- Compute (mod 2) $BCr = B(Cr) = x$

} Matrix-vector multiplication Takes $O(n^2)$ time

- If $x \neq y$, return NOT EQUAL o/w return EQUAL

Error Analysis If $BC = D$ (mod 2) ⇒ algorithm is always correct

If $BC \neq D$ (mod 2) ⇒ algorithm may fail
What is the probability of failure ?

Assume i^{th} row of BC and D are not equal

Let $u = i^{th}$ row of BC
$v = i^{th}$ row of D
. Then, $u \neq v$ by assumption

By previous lemma, $\mathbb{P}[\langle u,r \rangle \bmod 2 = \langle v,r \rangle \bmod 2] = \frac{1}{2}$

So, $\mathbb{P}[\text{fail}] \leq \frac{1}{2}$

We can make the error at most δ, by repeating $\log 1/\delta$ times

Random Variable

A random variable is a function $X : \Omega \longrightarrow V$

↳ value set

E.g. if $V = \mathbb{Z}$, X is a random integer
$V = \{0,1\}$, X is a random bit
$V =$ graph, X is a random graph

We write $\mathbb{P}[X=x]$ or $\mathbb{P}[X \leq x]$ or $\mathbb{P}[X=Y]$ to denote events about random variables

Expectation For real/complex/vector valued random variable X

$$\mathbb{E}[X] = \sum_x x \, \mathbb{P}[X=x]$$

E.g. $X =$ value of random dice $\quad \mathbb{E}[X] = \frac{7}{2}$

Note Random variables over infinite sample spaces (e.g. integers) may not have finite expectations

Conditional Expectation Given an event A, the conditional expectation of X given A is

$$\mathbb{E}[X \mid A] = \sum_x x \cdot \mathbb{P}[X=x \mid A]$$

$$\mathbb{E}[X] = \mathbb{E}[X \mid A] \cdot \mathbb{P}[A] + \mathbb{E}[X \mid \neg A] \cdot \mathbb{P}[\neg A]$$

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y=y] \cdot \mathbb{P}[Y=y] = \mathbb{E}\big[\mathbb{E}[X \mid Y]\big]$$

Independence Two random variables X and Y are independent if for all x, y: the events $X=x$ and $Y=y$ are independent

If X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Similarly, if $X_1, \ldots X_n$ are fully independent, then

$$\mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} \mathbb{E}[X_i]$$

may be dependent

Linearity For any random variables $X_1, \ldots X_n$ & reals $\alpha_1, \ldots \alpha_n$

$$\mathbb{E}\left[\sum_{i=1}^{n} (\alpha_i X_i)\right] = \sum_{i=1}^{n} \alpha_i \cdot \mathbb{E}[X_i]$$

Example Toss independent coins where each coin comes up heads w.p. $p \in [0,1]$
Count $\mathbb{E}[\#\text{ heads}]$

$$X_i = \begin{cases} 0 & \text{if coin is tails} \\ 1 & \text{if coin is heads} \end{cases} \quad \text{and} \quad \mathbb{E}[X_i] = p$$

Let $X = \sum_{i=1}^{n} X_i$

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = np$$

Example Toss independent coins where each coin comes up heads w.p. $p \in [0,1]$
How many flips until first head ?

$$\mathbb{E}[\#\text{ flips}] = \underbrace{\mathbb{E}[\#\text{ flips} \mid \text{first flip is heads}]}_{=1} \cdot \underbrace{\mathbb{P}[\text{first flip is heads}]}_{=p}$$

$$+ \underbrace{\mathbb{E}[\#\text{ flips} \mid \text{first flip is tails}]}_{= 1 + \mathbb{E}[\#\text{flips}]} \cdot \underbrace{\mathbb{P}[\text{first flip is tails}]}_{= 1-p}$$

$$= p + (1-p)(1 + \mathbb{E}[\#\text{flips}])$$

$$\Longrightarrow \mathbb{E}[\#\text{ flips}] = \frac{1}{p}$$

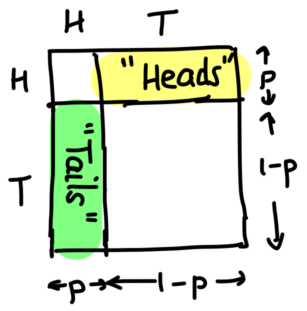
Sampling a Fair Coin from a Biased Coin

Suppose you have a biased coin that comes up heads with some unknown probability p
How can you use it to get a fair coin toss ?

Von Neumann in 1951 came up with a strategy

- Flip the biased coin twice
- If results of the two flips are different, return the first one
  HT → return "Heads" , TH → return "Tails"
- Otherwise repeat until success

Why does this return a fair coin toss ?



$$\mathbb{P}[HT] = \mathbb{P}[TH] = p(1-p)$$

So, $\mathbb{P}[HT \mid \text{flips diff}]$

$$= \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}$$

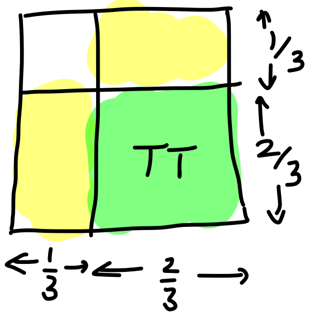
How many flips do we need ?

$\mathbb{P}$[each iteration succeeds] $= 2p(1-p) = q$ → How many times do we need to flip a biased coin until it comes up H ?

$\mathbb{E}$[# times until success] $= \frac{1}{q} = \frac{1}{2p(1-p)}$

Note There are better algorithms if we know the value of p

E.g. if $p = \frac{1}{3}$,



output "Heads" if we see TH or HT
"Tails" if we see TT

Las Vegas vs Monte Carlo Algorithms

So far we have seen two different types of randomized algorithms

Equality Testing Runs in a fixed polynomial time but small probability of error

Sampling a fair coin Runs in expected poly-time but zero-error

The first type of algorithm is called Monte Carlo, while the second one is called a Las Vegas algorithm

	Run time	Error
Monte Carlo	deterministic	probabilistic
Las Vegas	probabilistic	deterministic