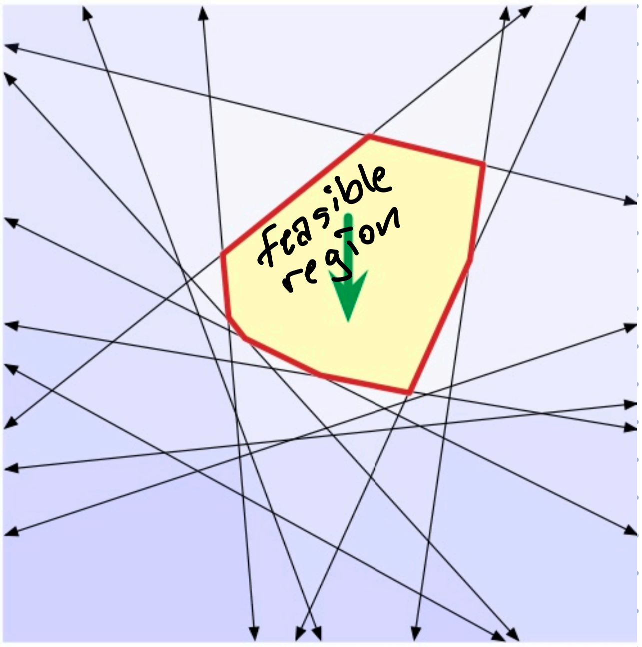
Linear programming — Find lowest point in a convex polyhedron



2 vars
n+2 constraints

Dual

n variables
n+2 constraints

d variables
n matrix constraints
d sign constraints

Primal (Π)

$$\begin{array}{ll} \max & c \cdot x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array}$$

$\Longleftrightarrow$

Dual ($\amalg$)

$$\begin{array}{ll} \min & y \cdot b \\ \text{s.t.} & yA \geq c \\ & y \geq 0 \end{array}$$

n variables
d matrix const
n sign const.

The Fundamental Theorem of Linear Programming. *A canonical linear program Π has an optimal solution x^* if and only if the dual linear program $\amalg$ has an optimal solution y^* such that* $\underbrace{c \cdot x^*}_{\text{objective values}} = y^*Ax^* = \underbrace{y^* \cdot b}$.

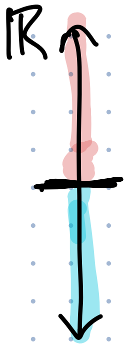
weak duality

Weak duality: IF x feasible for primal
y feasible for dual

Then $c \cdot x \leq yAx \leq y \cdot b$

Proof: x is Feasible $\Rightarrow Ax \leq b$
y is Feasible $\Rightarrow y \geq 0$
$yAx \leq yb$

primal-dual gap

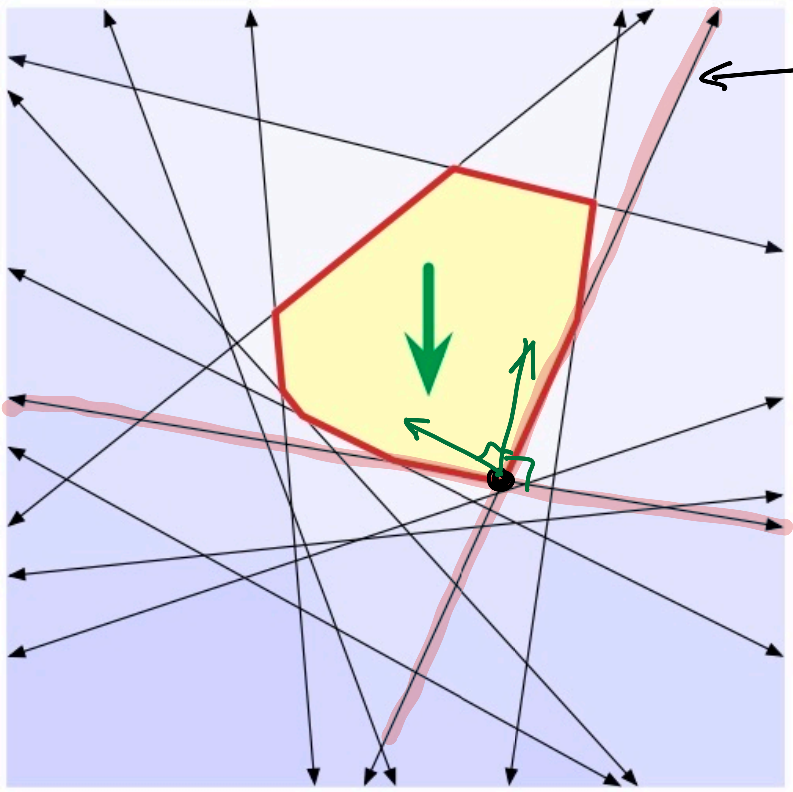


$\{y \cdot b \mid yA \geq c,\ y \geq 0\}$

Weak duality ⇒ no overlap

$\{c \cdot x \mid Ax \leq b,\ x \geq 0\}$

Strong ⇒ touch



tight = satisfied with equality

dual variables for optimal solution = coefficients of $-c$ in coord frame defined by tight primal constr

dual variable = primal const

dual 0 ⇔ primal constraint is loose

Basis = d ~~linearly independent~~ constraints

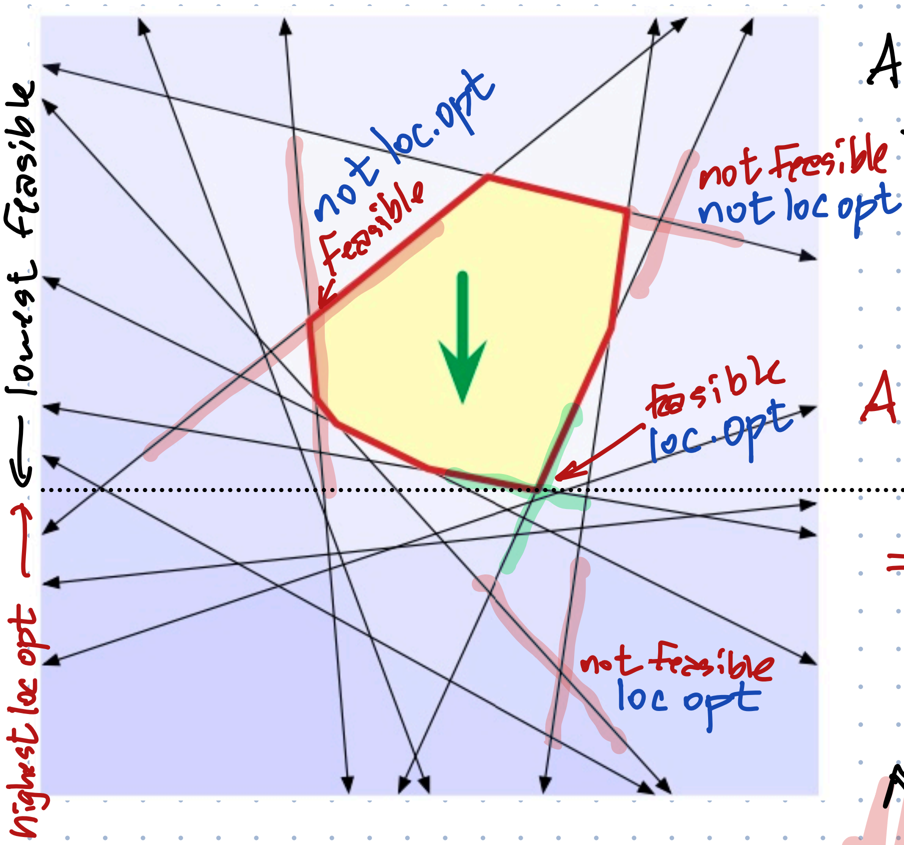
Assume no degeneracies

location of basis = point
= solution to system of eqns
= intersection of d constraint planes

value of basis = c · location

There are $\binom{n+d}{d} = \binom{d+n}{n}$ bases

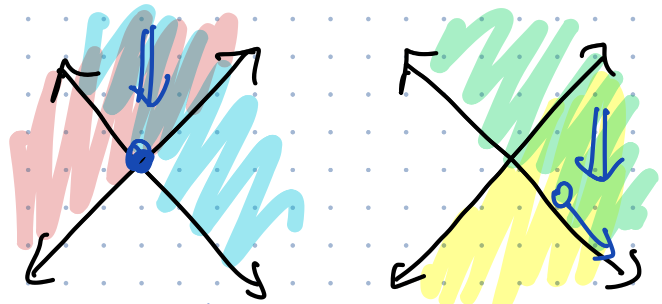
basis



A basis is feasible if $Ax \leq b$
$x \geq 0$
where x = location

A basis is locally optimal
= dual feasible
= optimal for LP containing only the constraints in the basis

optimal = feasible and loc. optimal



loc. opt.   not loc opt

primal-dual dictionary

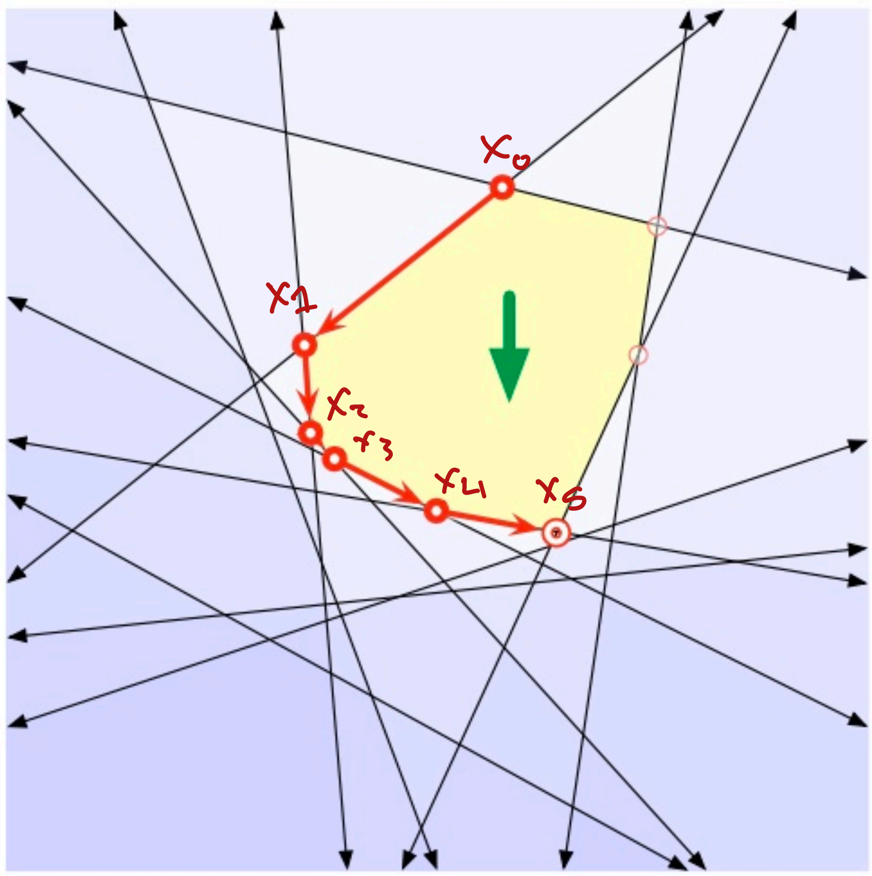
Primal	Dual
basis	(complement of) basis
value	value
$\binom{n+d}{d}$	$\binom{d+n}{n}$
Feasible	loc. opt.
loc. opt	Feasible
optimal	optimal
infeasible LP	unbounded LP
bounded LP	feasible LP

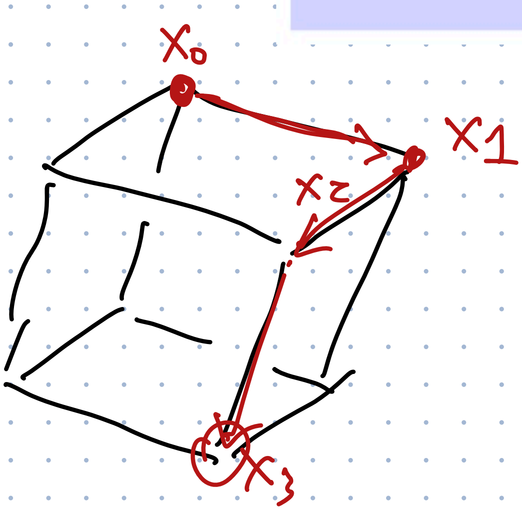
Basis B is a <u>neighbor</u> of basis B' if $|B \setminus B'| = 1$

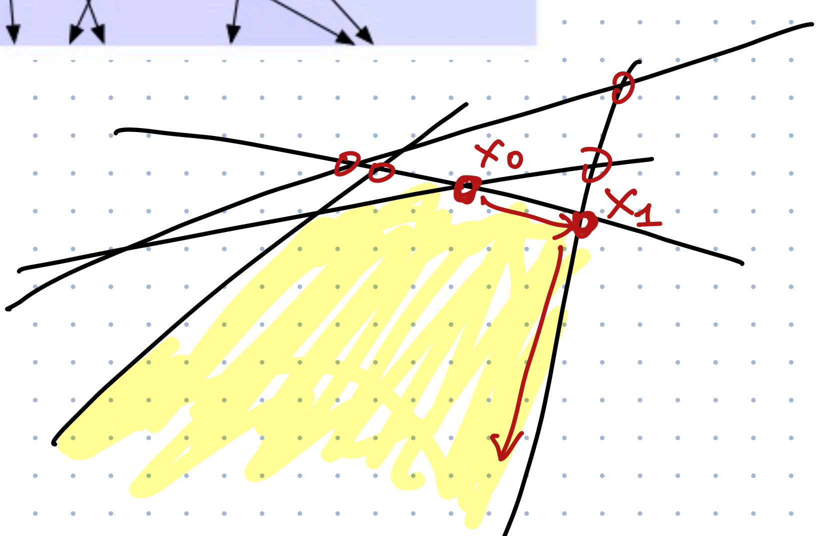
magic (for now)

```
PRIMALSIMPLEX(H):
  if ∩H = ∅
      return INFEASIBLE
  x ← any feasible vertex
  while x is not locally optimal
      ⟨⟨pivot downward, maintaining feasibility⟩⟩
      if every feasible neighbor of x is higher than x
          return UNBOUNDED
      else
          x ← any feasible neighbor of x that is lower than x
  return x
```

Feasibility doesn't depend on c.



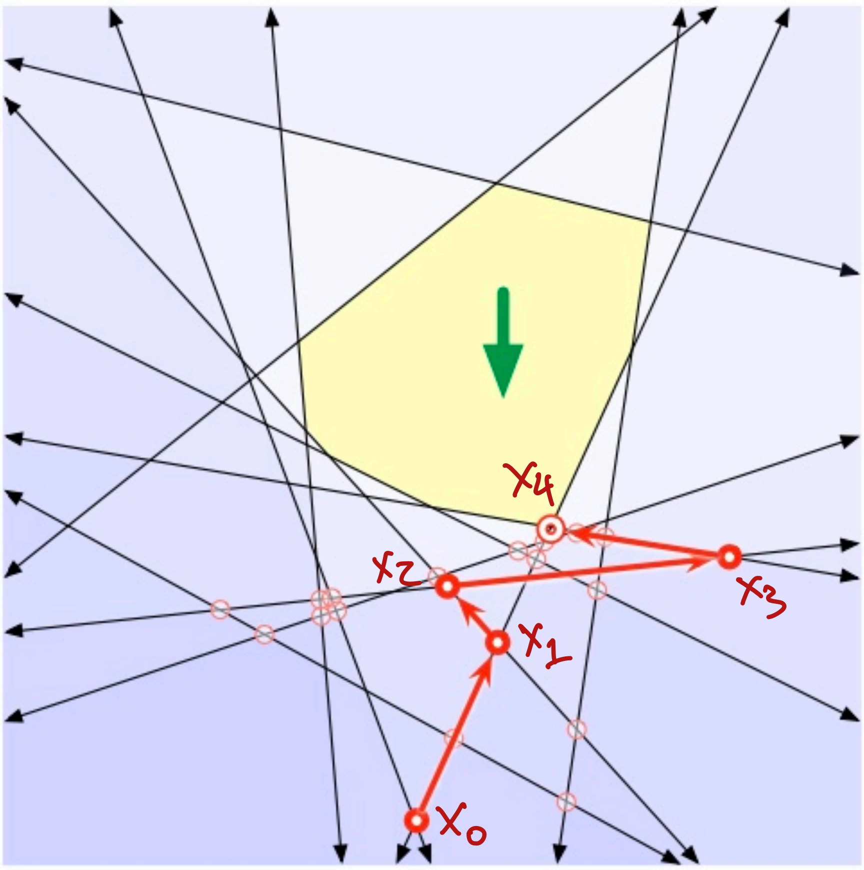


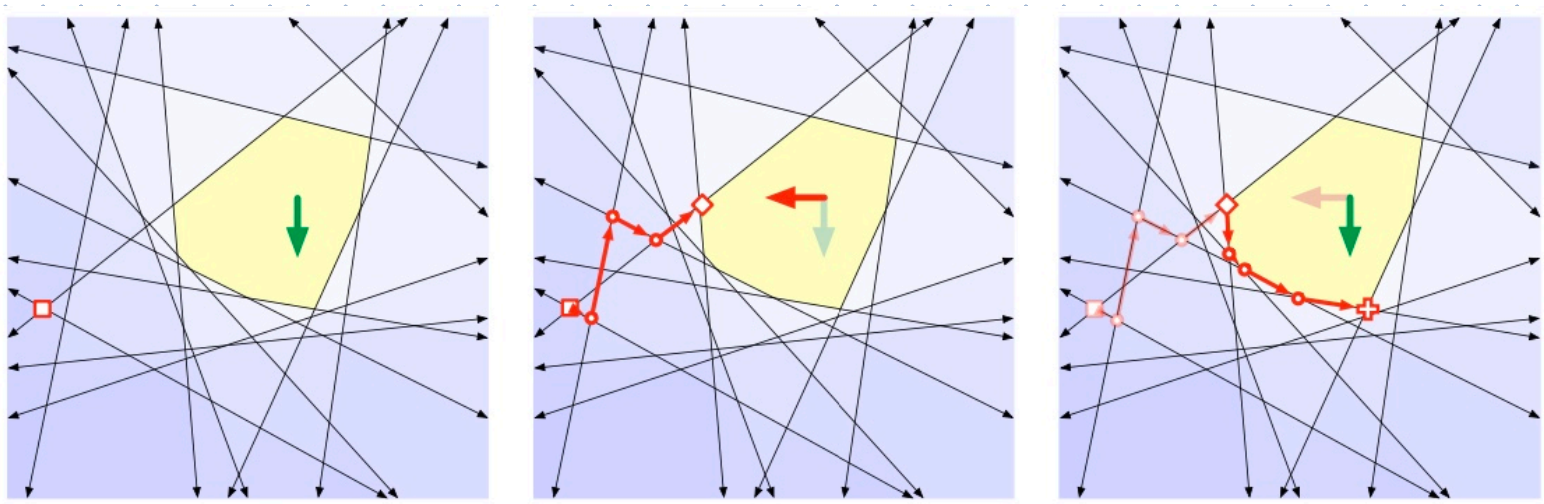


dual magic (for now)

DUALSIMPLEX(H):
- if there is no locally optimal vertex
  - return UNBOUNDED
- $x \leftarrow$ any locally optimal vertex
- while x is not feasbile
  - ⟨⟨*pivot upward, maintaining local optimality*⟩⟩
  - if every locally optimal neighbor of x is lower than x
    - return INFEASIBLE
  - else
    - $x \leftarrow$ any locally-optimal neighbor of x that is higher than x
- return x

loc opt doesn't depend on b





(a) (b) (c)

<u>DualPrimalSimplex(H):</u>

$x \leftarrow$ any vertex
$\tilde{H} \leftarrow$ any rotation of H that makes x locally optimal

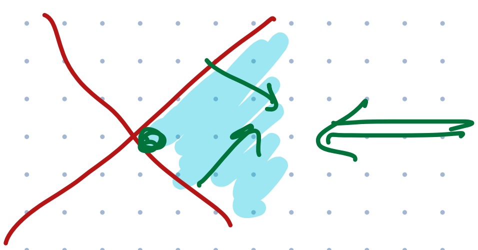
dual

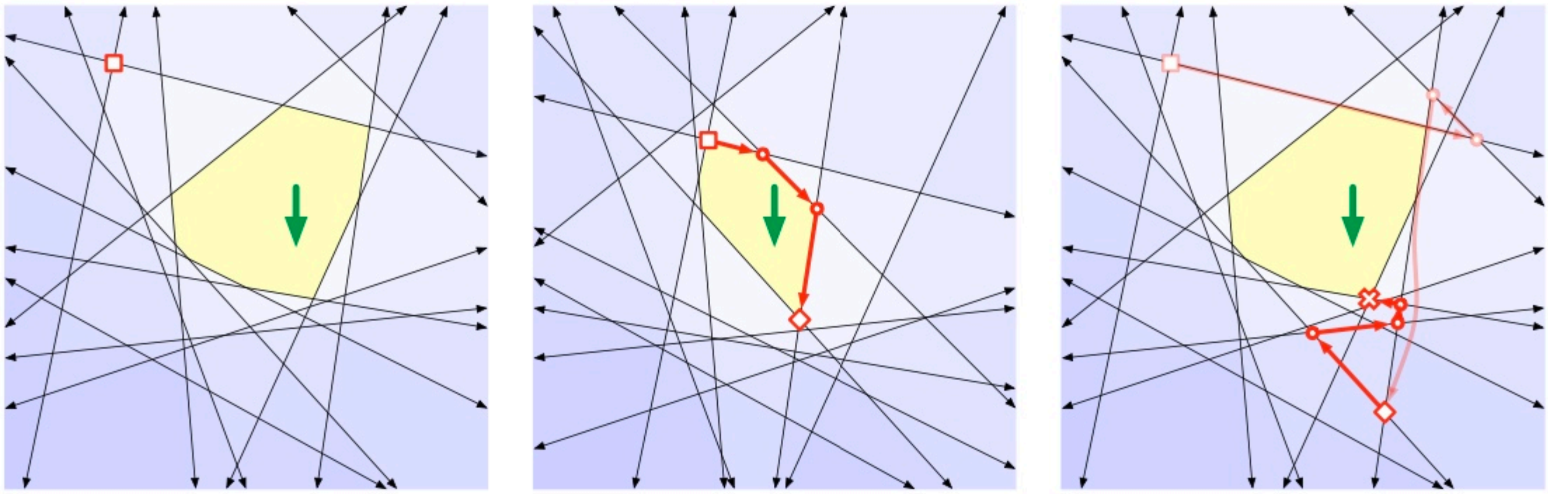
while x is not feasible
    if every locally optimal neighbor of x is lower (wrt $\tilde{H}$) than x
        return Infeasible
    else
        $x \leftarrow$ any locally optimal neighbor of x that is higher (wrt $\tilde{H}$) than x

primal

while x is not locally optimal
    if every feasible neighbor of x is higher than x
        return Unbounded
    else
        $x \leftarrow$ any feasible neighbor of x that is lower than x

return x





(a) (b) (c)

<u>PrimalDualSimplex(H):</u>

$x \leftarrow$ any vertex
$\tilde{H} \leftarrow$ any translation of H that makes x feasible

primal

while x is not locally optimal
  if every feasible neighbor of x is higher (wrt $\tilde{H}$) than x
    return Unbounded
  else
    $x \leftarrow$ any feasible neighbor of x that is lower (wrt $\tilde{H}$) than x

dual

while x is not feasible
  if every locally optimal neighbor of x is lower than x
    return Infeasible
  else
    $x \leftarrow$ any locally-optimal neighbor of x that is higher than x
return x