HW9 out later today → one week
HW10 out next Tue → due after break
"HW11" — practice

Suggestions for the last week?

Linear programming

Given $a_{ij}, b_i, c_j \in \cancel{\mathbb{R}}\ \mathbb{Z}$
Solve for variables $x_1, x_2 \ldots x_d$

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \le b_i \quad \text{for each } i = 1 \,..\, p$$

$$\sum_{j=1}^{d} a_{ij} x_j = b_i \quad \text{for each } i = p+1 \,..\, p+q$$

$$\sum_{j=1}^{d} a_{ij} x_j \ge b_i \quad \text{for each } i = p+q+1 \,..\, n$$

n constraints
d variables

Polytime algorithm if all input data is integral

Maximum flow

$$\text{maximize} \quad \sum_w f(s \to w) - \sum_u f(u \to s)$$

$$\text{subject to} \quad \sum_w f(v \to w) - \sum_u f(u \to v) = 0 \qquad \text{for every vertex } v \ne s, t$$

$$f(u \to v) \le c(u \to v) \qquad \text{for every edge } u \to v$$

$$f(u \to v) \ge 0 \qquad \text{for every edge } u \to v$$

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \le b_i \quad \text{for each } i = 1 \,..\, n$$

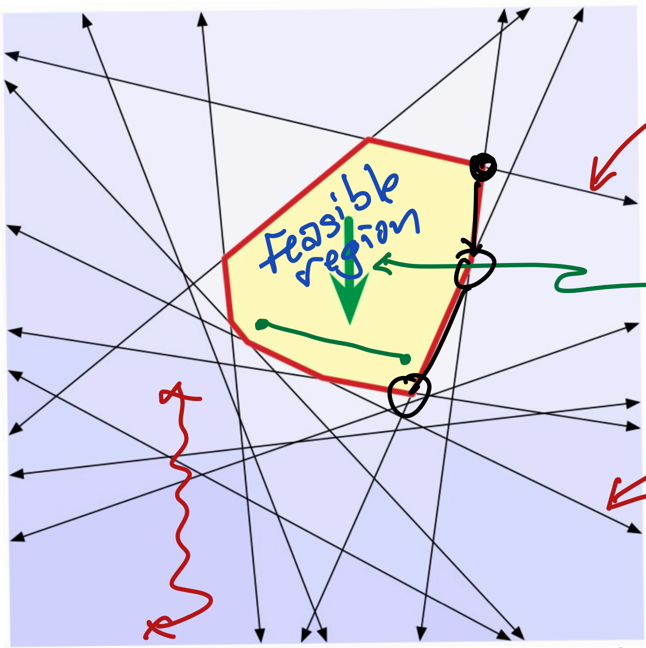
$$x_j \ge 0 \quad \text{for each } j = 1 \,..\, d$$

objective vector
unknown

$$\boxed{\begin{aligned} \max \quad & c \cdot x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}}$$

constraint matrix
offset vector

"canonical form"
"std inequality form"



$y \leq a_i x + b$

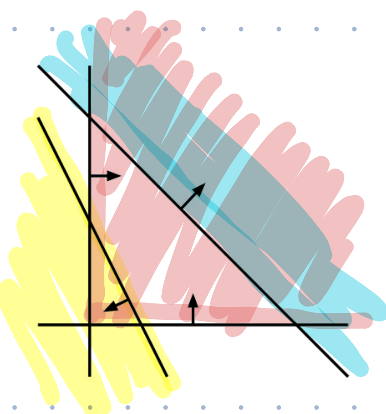
$\min y = 0 \cdot x + 1 \cdot y$

$y \geq a_j x + b$

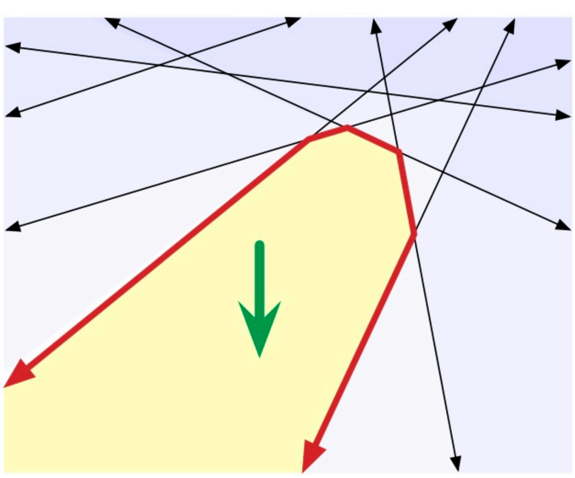
Feasible region = intersection of halfspaces
= convex polyhedron

LP = Find lowest point in convex polyhedron

$$
\begin{array}{ll}
\text{maximize} & x - y \\
\text{subject to} & 2x + y \leq 1 \\
& x + y \geq 2 \\
& x, y \geq 0
\end{array}
$$



Infeasible



Unbounded

Find the shortest path in a graph G from s to t.

variable = vertex
constraint = edge

$$
\begin{aligned}
\textbf{maximize} \quad & dist(t) \\
\text{subject to} \quad & dist(s) = 0 \\
& dist(v) - dist(u) \le \ell(u\to v) \quad \text{for every edge } u\to v
\end{aligned}
$$

u→v is ~~not~~ tense

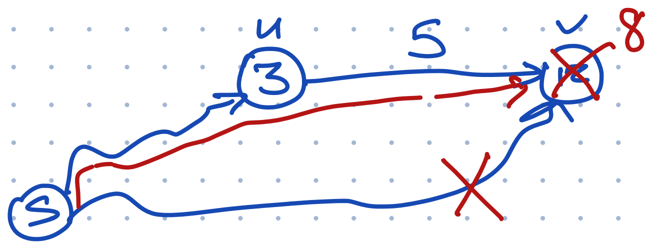
Ford's meta-algorithm:

Maintain v.dist at every node v

init: s.dist = 0 v.dist = ∞ ∀v ≠ s

Edge u→v is tense if

u.dist + ℓ(u→v) < v.dist

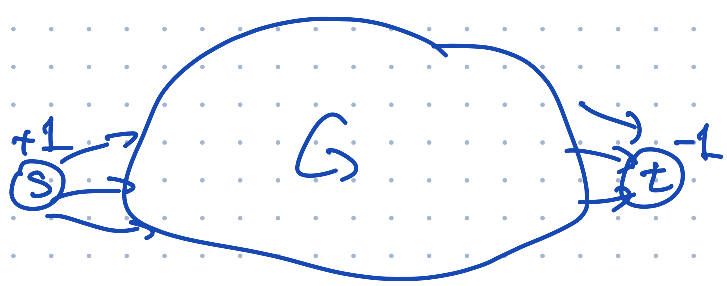


variable = edge
constraint = vertex

$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{u\to v} \ell(u\to v)\cdot x(u\to v) \\
\text{subject to} \quad & \sum_u x(u\to t) - \sum_w x(t\to w) = 1 \\
& \sum_u x(u\to v) - \sum_w x(v\to w) = 0 \quad \text{for every vertex } v \ne s,t \\
& x(u\to v) \ge 0 \quad \text{for every edge } u\to v
\end{aligned}
$$

Intuitively
$x(u\to v) = \begin{cases} 1 & \text{if } u\to v \text{ is on shortest path from } s \text{ to } t \\ 0 & \text{otherwise} \end{cases}$

Minimum cost flow!



Flow decomp ⟹ solution really is just a path

Primal (Π)

$$\begin{array}{ll} \max & c \cdot x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \end{array}$$

$\Longleftrightarrow$

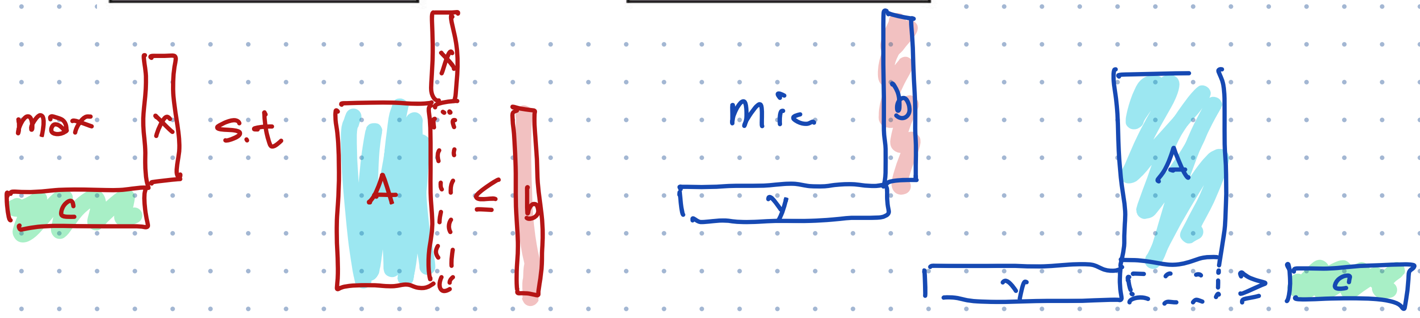
Dual ($\amalg$)

$$\begin{array}{ll} \min & y \cdot b \\ \text{s.t.} & yA \ge c \\ & y \ge 0 \end{array}$$

$\Longleftrightarrow$

Dual ($\amalg$)

$$\begin{array}{ll} \max & -b^\top \cdot y^\top \\ \text{s.t.} & -A^\top y^\top \le -c^\top \\ & y^\top \ge 0 \end{array}$$



The Fundamental Theorem of Linear Programming. *A canonical linear program Π has an optimal solution x^* if and only if the dual linear program $\amalg$ has an optimal solution y^* such that $c \cdot x^* = y^*Ax^* = y^* \cdot b$.*

Primal	Dual
$\max c \cdot x$	$\min y \cdot b$
$\sum_j a_{ij}x_j \le b_i$	$y_i \ge 0$
$\sum_j a_{ij}x_j \ge b_i$	$y_i \le 0$
$\sum_j a_{ij}x_j = b_i$	—
$x_j \ge 0$	$\sum_i y_i a_{ij} \ge c_j$
$x_j \le 0$	$\sum_i y_i a_{ij} \le c_j$
—	$\sum_i y_i a_{ij} = c_j$
$x_j = 0$	—

Primal	Dual
$\min c \cdot x$	$\max y \cdot b$
$\sum_j a_{ij}x_j \le b_i$	$y_i \le 0$
$\sum_j a_{ij}x_j \ge b_i$	$y_i \ge 0$
$\sum_j a_{ij}x_j = b_i$	—
$x_j \le 0$	$\sum_i y_i a_{ij} \ge c_j$
$x_j \ge 0$	$\sum_i y_i a_{ij} \le c_j$
—	$\sum_i y_i a_{ij} = c_j$
$x_j = 0$	—

$$\begin{aligned}
&\text{maximize} && dist(t) = \sum_v c_v \, dist(v) \\
&\text{subject to} && dist(s) = 0 \\
& && dist(v) - dist(u) \le \ell(u\to v) \quad \text{for every edge } u\to v
\end{aligned}$$

where $c_t = 1$
$c_v = 0$
for all $v \neq t$

variables: $dist(v)$ for each vertex
constraint = edge

A

$$E \; \underbrace{\begin{bmatrix} 0\,0\,0 + 1\,0\,0 - 1\,0\,0\,0 \\ \\ \end{bmatrix}}_{V}$$

$$A[u\to v, w] = \begin{cases} +1 & \text{if } w = v \\ -1 & \text{if } w = u \\ 0 & \text{o/o} \end{cases}$$

$$\begin{aligned}
&\text{minimize} && \sum_{u\to v} \ell(u\to v) \cdot x(u\to v) \\
&\text{subject to} && \sum_u x(u\to t) - \sum_w x(t\to w) = 1 \\
& && \sum_u x(u\to v) - \sum_w x(v\to w) = 0 \quad \text{for every vertex } v \neq s, t \\
& && x(u\to v) \ge 0 \quad \text{for every edge } u\to v
\end{aligned}$$

variable $x(u\to v)$ for every edge $u\to v$
constraints = vertices

$$A[u\to v, w] = \begin{cases} +1 & \text{if } v = w \\ -1 & \text{if } u = w \\ 0 & \text{o/w} \end{cases}$$