Team	Won–Lost	Left	NYY	BAL	BOS	TOR	DET
New York Yankees	75–59	28		3	8	7	3
Baltimore Orioles	71–63	28	3		2	7	4
Boston Red Sox	69–66	27	8	2		0	0
Toronto Blue Jays	63–72	27	7	7	0		0
Detroit Tigers	49–86	27	3	4	0	0	

$49+27=76$

Input: Wins[1..n] — past wins
Games[1..n, 1..n] — future games

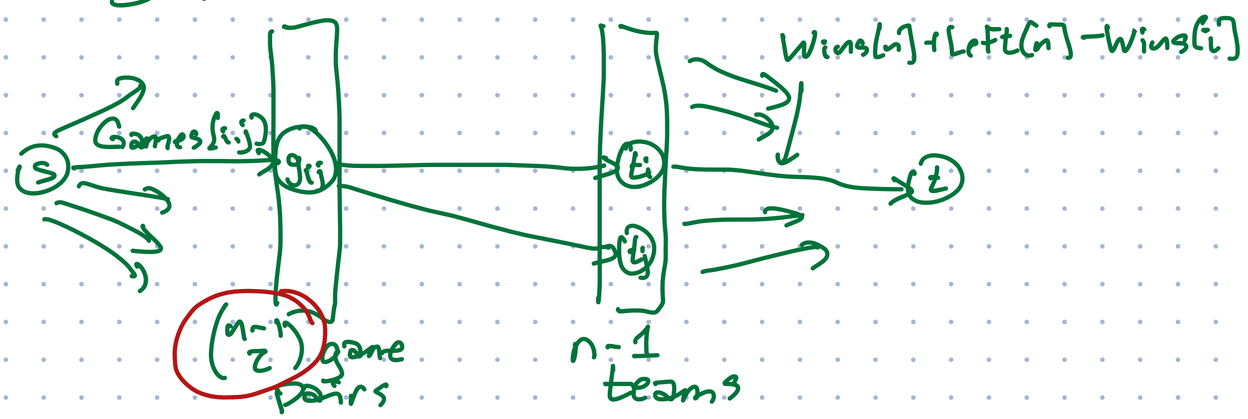
Output: True if team n could end season in 1st (maybe tied)
False o/w

$\text{Left}[i] = \sum_j \text{Games}[i,j]$

Assume team n wins all Left[n] games.

Team n wins season iff
every team i wins Wins[n]+Left[n]−Wins[i] future games

Build graph



We want to assign a winner to every game s.t. ....

Team	Won–Lost	Left	NYY	BAL	BOS	TOR	DET
New York Yankees	75–59	28		3	8	7	3
Baltimore Orioles	71–63	28	3		2	7	4
Boston Red Sox	69–66	27	8	2		0	0
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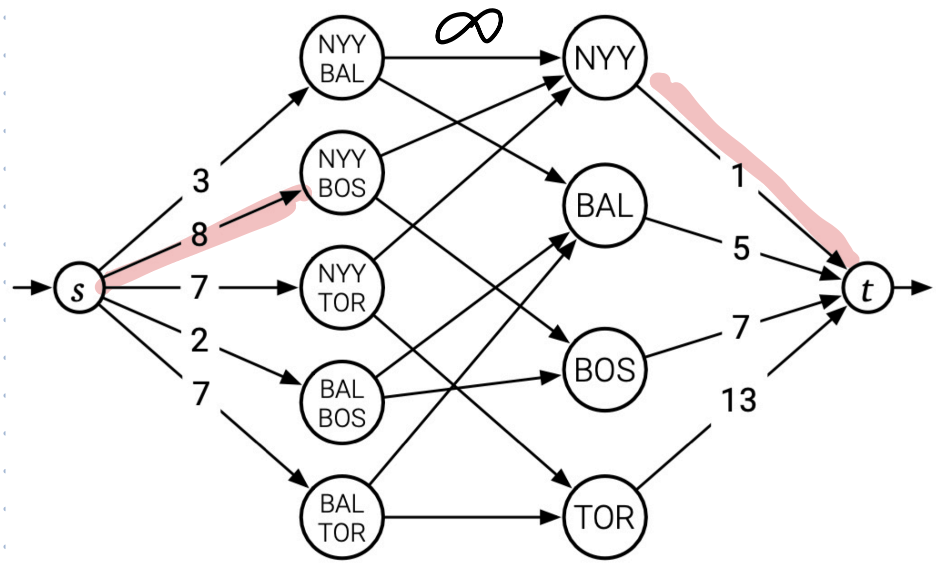


Thm:
Team n can win season ⟺
There is a max flow in G that saturates every edge out of s.

⇐ Suppose f flow saturates all edges out of s
Decompose f into paths
Each repenting one game
Cap $t_i \to t$ ⇒ no team overtakes team n ✓

⇐ Add 1 unit $s \to g_{ij} \to t_i \to t$
every time i beats j
Every game played ⇒ all $s \to g_{ij}$ are sat
Team n won ⇒ no edge $t_j \to t$ overflows

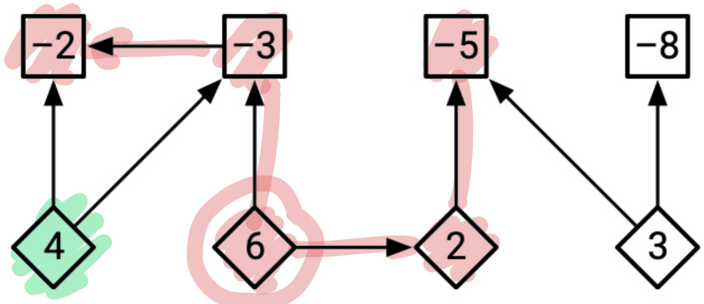
Algo: build G, compute max flow, check if all edges $s \to g$ saturated
$O(VE) = O(n^2 \cdot n^2) = O(n^4)$ time

Project selection / Open-pit mining

Input: n projects in a dag

$u \to v$ means u depends on v
v is a prereq for u
u can't start until v ends

$\$(v)$ profit
($\$(v) < 0$ means cost $-\$(v)$)



Output:
Subset S of projects
— "downward closed" ($u \in S$, $u \to v \in E$ $\Rightarrow$ $v \in S$)
— max $\sum_{v \in S} \$(v)$

Partition V into S – selected
T – turned down

min cut:
$\min \sum_{u \in S} \sum_{v \in T} c(u \to v)$

$$P = \sum_{\substack{v \in V \\ \$(v) \geq 0}} \$(v)$$

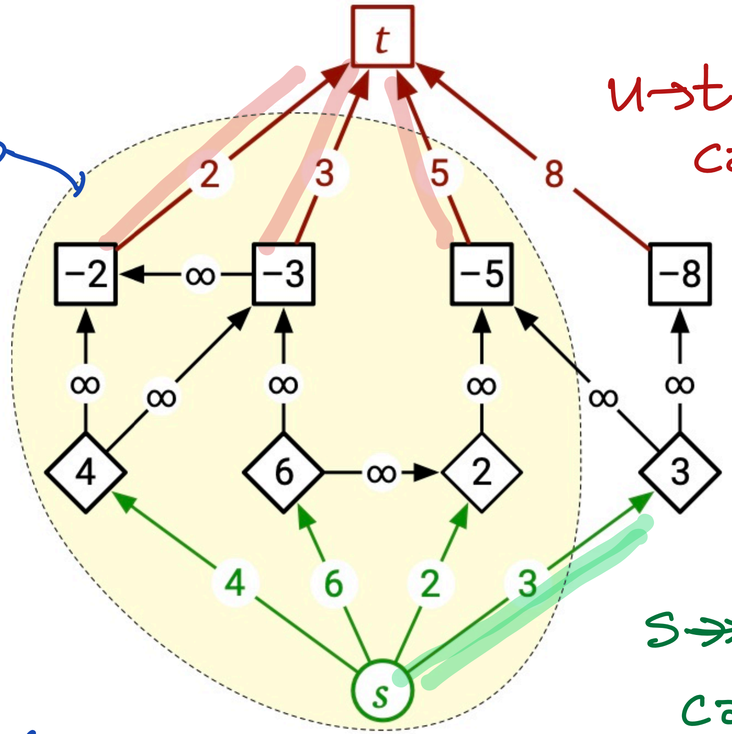
minimize

$$P - \sum_{v \in S} \$(v)$$

$\boxed{O(VE) \text{ time}}$

S

S is downward closed $\iff$ $\|S,T\|$ is finite



$u \to t$ iff $\$(u) < 0$
capacity $-\$(u)$

$s \to v$ if $\$(v) > 0$
capacity $\$(v)$

minimum cut (S,T)
maximizes $\sum_{x \in S} \$(x)$

$$\text{cost}(S) = \sum_{\substack{u \in S \\ \$(u) < 0}} -\$(u) = \sum_{u \in S} c(u \to t)$$

$$\text{income}(S) = \sum_{\substack{v \in S \\ \$(v) > 0}} \$(v) = \sum_{u \in S} c(s \to v)$$

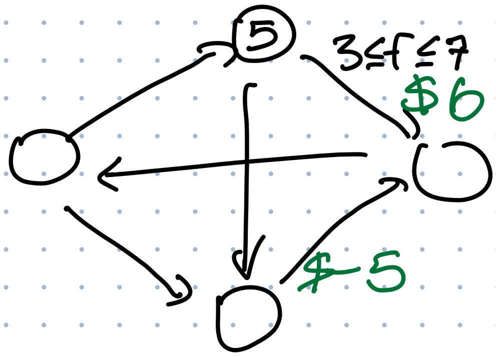
$$P = \text{income}(V) = \sum_{v} c(s \to v) = \text{income}(S) + \text{income}(T)$$

$$\text{profit}(S) = \text{income}(S) - \text{cost}(S)$$

$$\|S,T\| = \text{income}(T) + \text{cost}(S)$$

$$\|S,T\| = P - \text{profit}(S)$$

Minimum-cost flows



Input: dir. graph $G=(V,E)$

edges have capacity $c(e)$

lower bound $\ell(e)$

vertices have balances $b(v)$ — >0 demand, <0 supply

edge has cost $\$(e)$

Output: "Flow" f:

$\ell(u \to v) \le f(u \to v) \le c(u \to v) \quad \forall u \to v$

$\sum_u f(u \to v) + b(v) = \sum_w f(v \to w) \quad \forall v$

$\min \sum_{u \to v} \$(u \to v) \cdot f(u \to v)$

$O(E^2 \log V)$ time