HW 7 out due next Wed — last before MT2

MT2 week from Monday

# Maximum Flows / minimum cuts

Input: "Flow network" — directed graph $G=(V,E)$

two vertices $s, t$

capacities $c(e) \geq 0$

**Max Flow:**

compute $f: E \to \mathbb{R}_{\geq 0}$
$$\begin{cases} f(u \to v) \geq 0 \\ f(u \to v) \leq c(u \to v) \\ \sum_v f(u \to v) = \sum_v f(v \to u) & \text{for all } u \neq s,t \end{cases}$$

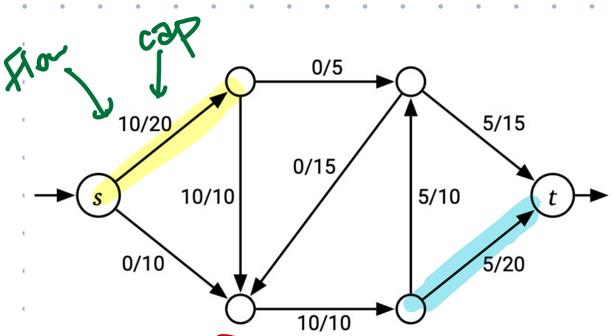
maximize $|f|$
$$= \sum_w f(s \to w) - \sum_u f(u \to s)$$
$$= \sum_u f(u \to t) - \sum_w f(t \to w)$$

**Min cut:**

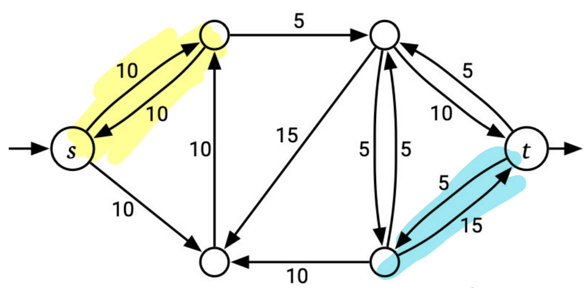
Compute partition $V = S \sqcup T$ $\quad S \cap T = \emptyset$, $S \cup T = V$

minimize $\|S,T\| = \sum_{u \in S} \sum_{v \in T} c(u \to v)$

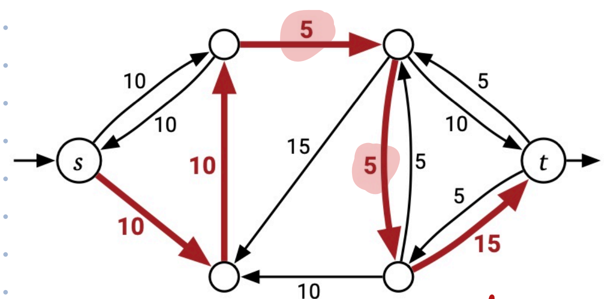
Max flow - mincut theorem: $|f| = \|S,T\|$



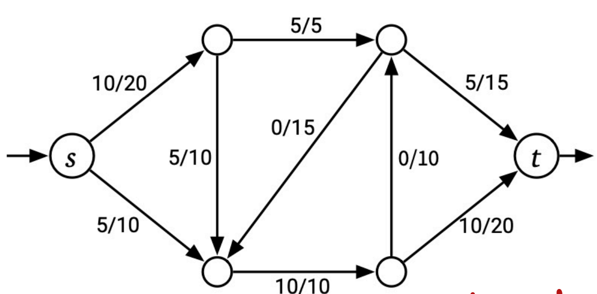

flow network




residual graph




augmenting path



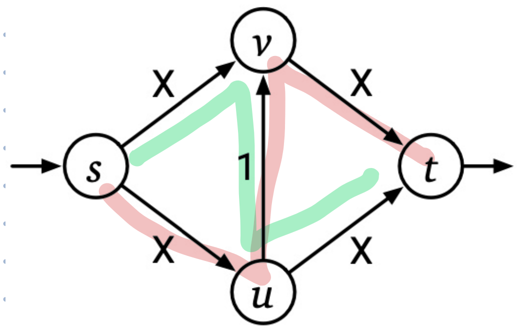

updated flow network

$\boxed{\text{IF}}$ this algo halts, it returns max flow.

Integrality: If all caps are integers, FF always returns integer maxflow in time $O(E \cdot |f^*|)$

$f^*$ = max flow

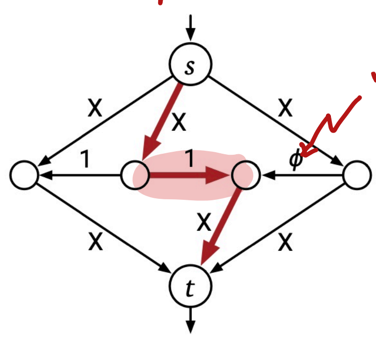
Input can be encoded using $O(\log X)$ bits



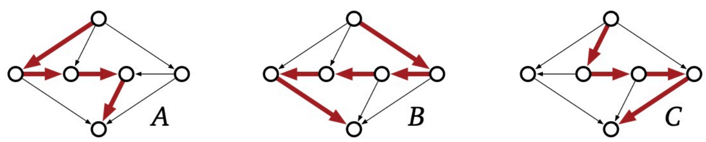

max flow value = $2X$

But FF can use $2X$ iterations

IF we have irrational capacities FF may $\infty$ loop




$\frac{\sqrt{5}-1}{2} \approx 0.618\ldots$




Augment along $\not{}$ B C B A B C B A . . . .

$1 \; \phi \; \phi \; \phi^2 \; \phi^2 \; \phi^3 \; \phi^3 \ldots$

$\text{total flow} \leq 1 + 2\sum_{i \geq 1} \phi^i < 7$

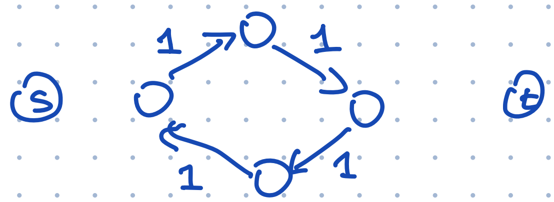
$\text{Max Flow} = 2X + 1 \gg 7$

| |
|---|
| For purposes of HWs and exams: Max Flow solved in $O(VE)$ time |

# Flow decomposition

- Flow = function on edges satisfying some constraints
- Flow = sum of paths and cycles

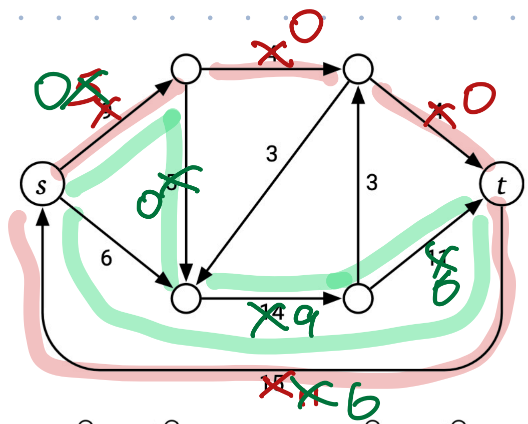

paths → Flows



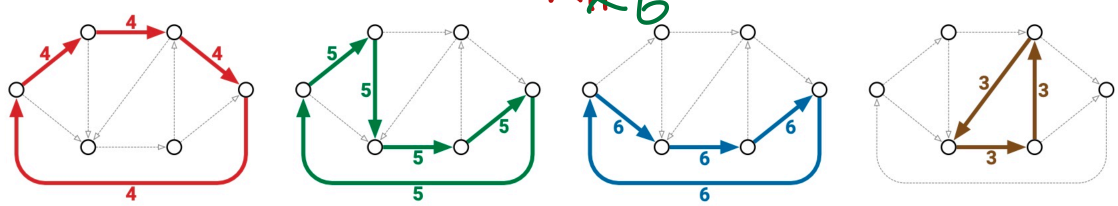

cycles → Flows

Two Flows F and F' ⟹ F+F' is a flow

⟹ $\alpha \cdot F$ is a flow for any $\alpha > 0$




Circulation = Flow with value 0




Any circulation can be decomposed into ≤ E cycles in O(VE) time using O(VE) space

- Any flow is weighted sum of paths & cycles using only ~~forward~~ edges
- Acyclic flow ⟺ only paths
  Acyclic max flow!
- |F| = sum of weights
- Flow is integral → weights are integral

Edmonds-Karp heuristics:

1. Choose fattest augmenting path
   integer capacities $O(E^2 V \log |f^*|)$

2. Shortest augmenting path $O(E^2 V)$
   min#edges

Faster! Faster! Kill! Kill!

Orlin 2012 $O(VE)$

Chen 2022 $O(E^{1+o(1)} \log U)$ max integer capacity

Bernstein 2024 $O(V^{2+o(1)} \log U)$
augmenting path