L22 Approx via LP

11/7/23

G

$\forall v \in V \quad w_v \geq 0$ cost

Compute VC of min cost

write it as an IP.

$$\boxed{\begin{aligned} &\min \sum_{v\in V} x_v w_v \\ &\forall v \in V \quad x_v \in \{0,1\} \\ &\forall uv \in E(G) \quad x_u + x_v \geq 1 \end{aligned}}$$

LP

$$\boxed{\begin{aligned} &z = \min \sum_{v\in V} x_v w_v \\ &\forall v \in V \quad x_v \geq 0 \\ &\forall uv \in E \quad x_u + x_v \geq 1 \end{aligned}}$$

Solve the LP

$\hat{x}_v$ value of x_v in the LP solution

$\hat{z}$ value of the LP

$$\text{cost}(\text{opt VC}) \geq \hat{z}$$

$$U = \{v \in V \mid \hat{x}_v \geq \tfrac{1}{2}\}$$

why U is a VC?

$uv \in E \quad \hat{x}_u + \hat{x}_v \geq 1$

$\Rightarrow \hat{x}_u \geq \frac{1}{2}$ or $\hat{x}_v \geq \frac{1}{2}$

$\Rightarrow u$ or v are in U.

$$\text{cost}(U) = \sum_{u\in U} w_u \leq \sum_{u \in U} 2\hat{x}_u w_u = 2\sum_{u\in U} \hat{x}_u w_u$$
$$\leq 2 \underline{\underline{\sum_{v \in V}}} \hat{x}_v w_v = 2\hat{z} \leq 2(\text{opt}\overset{\min}{} \text{cost VC})$$

Set cover

$(U, F) \quad n = |U| \quad F = \{f_1, \dots, f_m\}$, $f_i \subseteq U$

Q: Find min size $G \subseteq F$ s.t.

$\bigcup G = \bigcup_{f \in G} f = U$ (everything is covered)

min $|G|$

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k sets opt solution $\Rightarrow$ greedy alg output a cover of size $\lceil k \ln n \rceil + 1$

$\Theta(\log n)$-approx

$$\frac{\text{size approx solution}}{\text{opt solution size}} \Leftarrow \text{approx. ratio}$$

Weighted set cover

$(U, F) \quad F = \{f_1, f_2, \dots, f_m\}$

$U = \{p_1, p_2, \dots, p_n\}$

$c_i \geq 0$ cost of f_i

Q: Compute set cover $G \subseteq F$ of min total cost.

LP set cover

$$z = \min \sum x_i c_i \qquad c_i \text{ cost of } f_i$$

$p_i \quad \forall i \quad \sum_{\ell : (f_\ell \ni p_i)} x_\ell \geq 1$

$\forall i \quad x_i \geq 0$

Solve the LP $\hat{x}_i$ fractional value

$G_1 \equiv$ choose set f_i with probability $\hat{x}_i$

G_2

G_3

$\vdots$

$G_t = \quad t = \quad \log n)$

Solution $\bigcup_{i=1}^{t} G_i$

$$E[\text{cost}(G_1)] = E\left[\sum_{i=1}^{m} Y_i c_i\right] = \sum_{i=1}^{m} E[Y_i c_i] = \sum c_i E[Y_i]$$
$$= \sum_{i=1}^{m} c_i \hat{x}_i = \hat{z} \equiv \text{lp value}$$

$Y_i = 1 \iff f_i \in G_1$

$$E\left[\text{cost}\left(\bigcup G_i\right)\right] \leq t \cdot \text{LP\_value} \leq t \cdot (\text{cost of opt solution})$$

$p_i \qquad \hat{x}_1 + \hat{x}_2 + \hat{x}_3 + \dots \hat{x}_\alpha \geq 1$

$$\beta = P\left[p_i \text{ is not covered by } G_1\right] = (1 - \hat{x}_1)(1-\hat{x}_2)$$
$$\cdots (1 - \hat{x}_u) \leq \exp\Big(\underbrace{\overbrace{-\hat{x}_1 - \hat{x}_2 - \cdots - \hat{x}_\alpha}^{\leq -1}}_{\hat{x}_1 + \hat{x}_2 + \dots + \hat{x}_\alpha \geq 1}\Big) \leq \frac{1}{e}$$

$$1 - x \leq e^{-x} = \exp(-x)$$

$$P\left[p_i \text{ is not covered by } G_1, G_2, \dots, G_t\right] = P\left[G_1 \text{ not covering } p_i \text{ and } G_2 \text{ and } G_3\right]$$
$$= \prod_{\ell=1}^{t} P[G_\ell \text{ not covering } p_i]$$
$$= \prod_{\ell=1}^{t} \beta = \beta^t \leq \frac{1}{e^t} \leq \frac{1}{n^{10}} \qquad t = 10 \ln n$$

$$P\left[\text{any element is not covered}\right] \underset{\text{union bound}}{\leq} \sum_{i=1}^{n} P[p_i \text{ is not covered}]$$
$$\leq \sum_{i=1}^{n} \frac{1}{n^{10}} = \frac{1}{n^9}.$$

Thm weighted set cover can be approximated using LP rounding with $O(\log n)$-approximation.

Graph congestion

G: Graph (directed)

$\forall i = 1, \dots, u \quad s_i, t_i$ send packet from s_i to t_i

π_i and σ_i two paths from s_i to t_i.

Minize congestion.

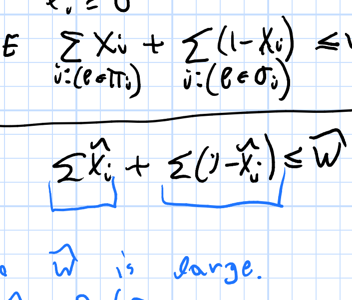
Given a solution

for an edge e, let

$\text{load}(e) \equiv$ # paths using e

Q: find assignment that minimizes

$$\text{cong}(G) = \max_{e \in E} \text{load}(e)$$



$$\boxed{\begin{aligned} &\min \ w \\ &\forall i \quad x_i \geq 0 \\ &\forall e \in E \quad \sum_{i:(e \in \pi_i)} x_i + \sum_{i:(e \in \sigma_i)} (1 - x_i) \leq w \end{aligned}}$$

specific e $\qquad \sum \hat{x}_i + \sum (1 - \hat{x}_i) \leq \hat{w}$

Assume $\hat{w}$ is large.

$\hat{w} = \Omega(\sqrt{n})$

$Y = \sum_{i=1}^{n} X_i \qquad X_i \in \{0, 1\} \Leftarrow$ indicator variables

$\mu = E[Y]$

Chernoff's inequality

$$P[Y \geq (1+\delta)\mu] \leq \exp\left(-\frac{\mu\delta^2}{4}\right)$$

$\hat{w} = \Omega(\sqrt{n})$