L19: Approximation alg III

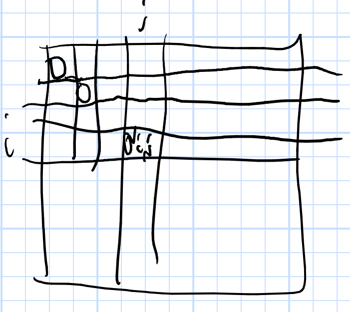
10/26/2023

$TSP_{\le}$ TSP with triangle inequality

n cities; $V=[n]$

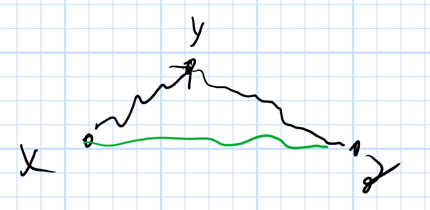
$\forall i,j \quad d_{i,j}$ = distance between i and j

D



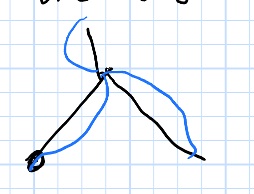
K_n = clique of size n

triangle inequality



$\forall x,y,z \in V \qquad d(x,y)+d(y,z) \ge d(x,z)$

- Compute MST $T=MST(K_n)$
- Duplicate all the edges in T, $\Rightarrow H$



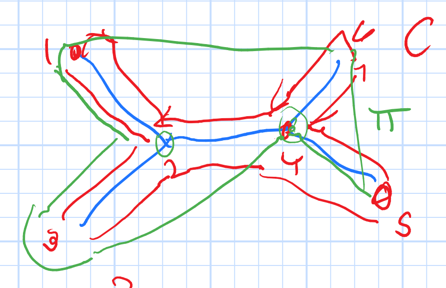
$d_T(v) \qquad d_H(v)=2d_T(v) \Leftarrow$ even

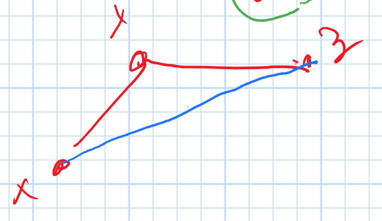
H is connected, and all the vertices have even degree

$\Rightarrow$ Eulerian cycle $\exists$ in H

$\Rightarrow$ Compute an Eulerian cycle C for H.

$\Rightarrow$ "output C as the TSP tour"



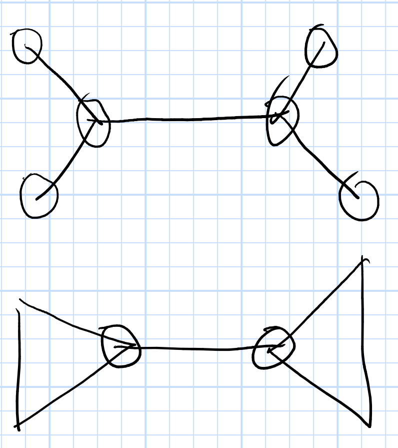


$cost(\pi) \le cost(H) = 2cost(MST) \le 2cost(Opt\,TSP)$

Theorem

G: n, vertices m edges in $O(n\log n+m)$ time the alg computes a TSP tour π s.t. $cost(\pi) \le 2cost(OptTSP)$.

3/2-approx to $TSP_{\le}$ (Yannakakis)

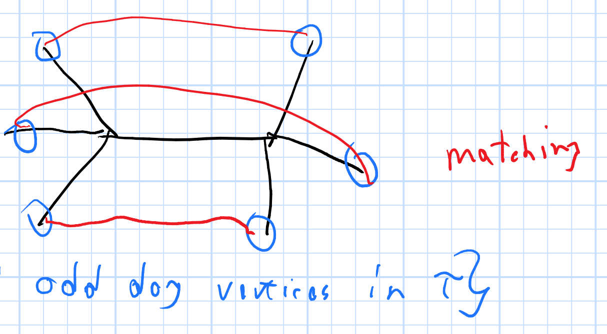


Observation

# of odd degree vertices in the graph is even

proof $\underbrace{\sum_{v\in V} d(v)}_{\#\text{ odd term must be even}} = \underbrace{2m}_{\text{even number}}$

T: compute the MST



U = {all odd deg vertices in T}

K_U = clique over U

M = Compute minimum weight perfect matching

$K = T \cup M$.

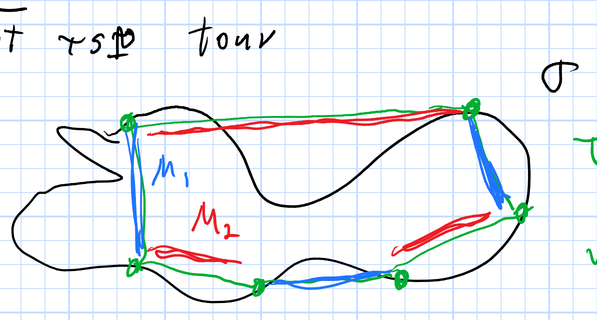
$C \Leftarrow$ Eulerian cycle of K

$\pi \Leftarrow$ shortcut C into a TSP tour.

Claim $w(M) \le w(Opt\,TSP)/2$

proof

σ: opt TSP tour



$\tau = \sigma$ shortcut on U.

$w(\tau) \le w(\sigma)$

$w(M_1)+w(M_2) = w(\tau) \le w(OptTSP)$

assume $w(M_1) \le w(M_2) \Rightarrow$

$2w(M_1) \le w(M_1)+w(M_2) \le w(opt\,TSP)$

$w(M_1) \le \frac{w(Opt\,TSP)}{2}$ ∎

Clustering

V = set of n elements

d is a metric of V:

- $\forall v\in V \quad d(v,v)=0$
- $\forall u,v \quad d(u,v)=0 \Rightarrow u=v$.
- $\forall x,y,z \quad d(x,y)+d(y,z) \ge d(x,y)$.

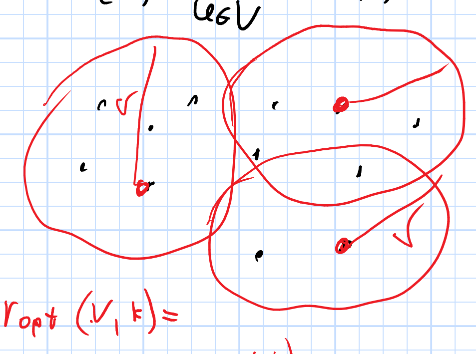
k-center clustering

$C \subseteq V$ a set of k centers

$\forall u \in V \qquad d(u,C) = \min_{c\in C} d(u,c)$

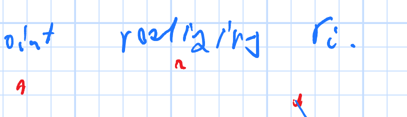
radius of clustering of C is

$r(C) = \max_{u\in V} d(u,c)$



$r = r(C)$

$r_{opt}(V,k) = \min_{\substack{C\subseteq V\\ |C|=k}} r(C) = \min_{\substack{C\subseteq V\\ k=|C|}} \max_{u\in V} d(u,C)$

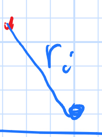


$C_i = \{c_1, c_2, \ldots, c_i\}$

$r_i = r(C_i)$ = radius for this set of centers

c_{i+1} is the point realizing r_i.

$O(nk)$ time.



$r(C_k) = r_k \le 2r_{opt}(V,k)$

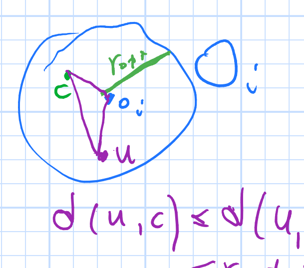
$r_1 \ge r_2 \ge r_3 \ge \cdots \ge r_k$

- $o_1, o_2, \ldots, o_k$ optimal centers.

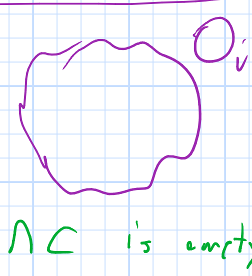
$O_\ell = \{v\in V \mid d(o_\ell, v) \le r_{opt}\}$

$\bigcup_{\ell=1}^{k} O_\ell = V$

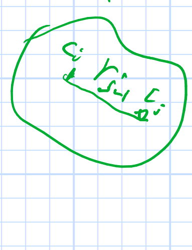
- $\forall O_i$ exist center $\underset{\substack{\Downarrow\\ \text{set computed}}}{c\in C}$ $\quad c \in O_i$



$\forall u \in O_i \quad d(u,c) \le d(u,o_i) + d(o_i,c)$
$\le r_{opt} + r_{opt} \le 2r_{opt}$.



$O_1, \ldots, O_k$
$c_1, \ldots, c_k$

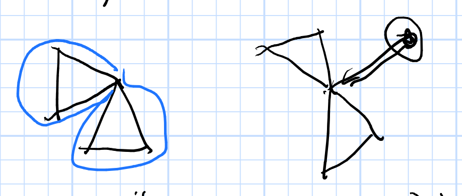


if $O_i \cap C$ is empty.

2-approx!

Eulerian cycle

C is an Eulerian cycle of G if C visits every <u>edge</u> of G exactly once.



Thm G is Eulerian $\Longleftrightarrow$ G is connected and all vertice have even degree.

Proof