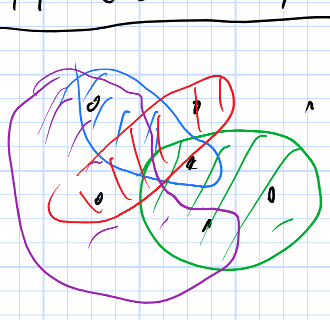
L18 10/24/23

approximation alg II



$\boxed{(U, F)}$ set system
Hypergraph

$n = |U|$ $F = \{f_1, f_2, \dots, f_m\}$

$f_i \subseteq U$

Set cover

Given (U, F) Compute minimal number t of of subsets

$g_1, g_2, \dots, g_t \in F$

$U = \bigcup_{i=1}^{t} g_i$

Approx alg for set cover.

while U is not covered do:

Compute set g_i of that covers largest number of elements not covered yet

$C \leftarrow C \cup \{g_i\}$

Lemma A Greedy alg provides $O(\log n)$ approx to set cover. $n = |U|$ (U, F): instance

$\Rightarrow$ # of sets output is $O(k \log n)$

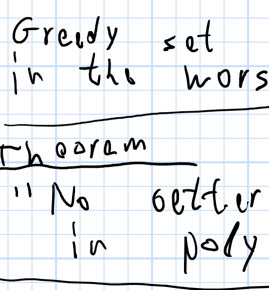
$\uparrow$ # set in opt solution.

Lemma B (*)

Greedy set cover is $\Omega(\log n)$ approx in the worst case.

Theorem

"No better approx is possible in poly time."



n elements

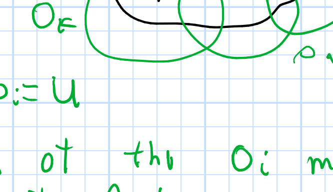
$\Omega(\log n)$ sets

$\boxed{\text{QED B}}$

Proof Lemma A

$n_i :=$ # elements not covered in the beginning of ith iteration

$n_1 = n$



$\bigcup_{i=1}^{k} O_i = U$

one of the O_i must cover at least n_i/k elements.

$\Rightarrow g_i$ covers at least $\frac{n_i}{k}$ new elements.

$$n_{i+1} \le n_i - \left|\begin{matrix}\text{\# elem} \\ \text{covered} \\ \text{by } g_i\end{matrix}\right| \le n_i - \frac{n_i}{k} \le \left(1 - \frac{1}{k}\right) n_i$$

$\rotatebox{90}{$\le$}$ $\frac{n_i}{k}$?

$$n_{i+1} \le \left(1 - \frac{1}{k}\right) n_i \le \left(1 - \frac{1}{k}\right)^2 n_{i-1} \le \left(1 - \frac{1}{k}\right)^i n$$

$$\le \left(\exp\left(-\frac{1}{k}\right)\right)^i n = \exp\left(-\frac{i}{k}\right) n$$

$i = \lceil k \ln n \rceil + 1$ $\quad \le \exp\left(-\frac{\lceil k \ln n \rceil + 1}{k}\right) \cdot n$

< 1

$\Downarrow$

$n_{i+1} = 0$

$O \subseteq F$

$\rotatebox{90}{$=$}$

$\{O_1, \dots, O_k\}$

$\frac{n_i}{k}$

Lemma $x \ge 0$ $1 - x \le e^{-x} = \exp(-x)$

$$n_{k+1} \le \left(1 - \frac{1}{k}\right)^k n \le \frac{n}{e}$$

$$\left(1 - \frac{1}{k}\right)^k \le \frac{1}{e} \le \left(1 - \frac{1}{k}\right)^{k-1} \quad \forall k$$

2SAT

$\underbrace{(x_i \vee \bar{x}_j)}_{\text{clause}} \wedge (x_2 \vee \bar{x}_{17}) \dots (x_{17} \vee \bar{x}_{13})$ n vari, m clauses

Q: Is there a satisfying assignment?

2SAT can be solved in linear time $O(n+m)$.

$x_1 = 0$ $\quad x_2 = 0$ $\quad x_7 = 0$ $\quad (x_7 \vee x_{19})$

$x_1 = 1$ $\quad x_2 = 1$

$x_7 = 0 \Rightarrow x_{19} = 1$

$x_{19} = 0 \Rightarrow x_7 = 1$

$(\bar{x} \vee y)$

$\rotatebox{90}{$=$}$

0

Exact3SAT

$(x \vee \bar{y} \vee z) \wedge (\quad) \dots (\quad)$

3SAT

Given Exact3SAT formula is there a sat assignment?

3SAT is NP-hard

Max 3SAT

Given an Exact 3SAT formula compute an assignment that satisfies as many clauses as possible.

n variables $x_1, x_2, \dots, x_n$

m clauses $C_1, C_2, \dots, C_m$

Alg

Randomly assign 0/1 with equal probability to the variables.

Analysis

$Y_i = 1 \Leftrightarrow$ The random assignment sat the ith clause $C_i \equiv (x \vee y \vee z)$

$$P[Y_i = 1] = \frac{7}{8}$$

linearity of expectation

$$E\left[\begin{matrix}\text{\# of} \\ \text{clauses}\end{matrix} \text{ sat}\right] = E\left[\sum Y_i\right] \overset{\downarrow}{=} \sum_{i=1}^{m} E[Y_i]$$

$$= \sum_{i=1}^{m} \frac{7}{8} = \frac{7}{8} m$$

Thm Unless $P = NP$ no better approx to Max 3SAT is possible in polynomial time.

Max 2SAT is NP-hard

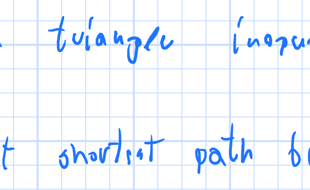
weighted matching

$G = (V, E)$ weights on the edges

Q: Compute a matching of maximum weight

Can be solved in polynomial time.

TSP problem



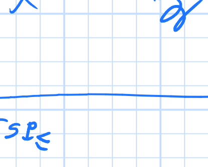
TSP with triangle inequality

$G = (V, E)$

$d(u, v) =$ len of shortest path between u and v

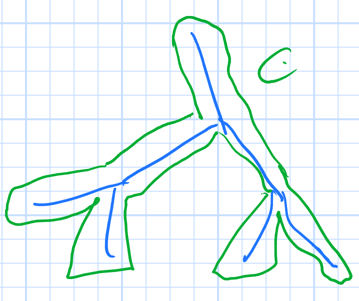
G is a clique

$\forall x, y, z$ $\quad w(xy) + w(yz) \ge w(xz)$



Approx to $TSP_\le$

- Compute MST
- $w(MST) \le w(\text{optimal TSP tour})$



$w(C) = 2 w(MST)$