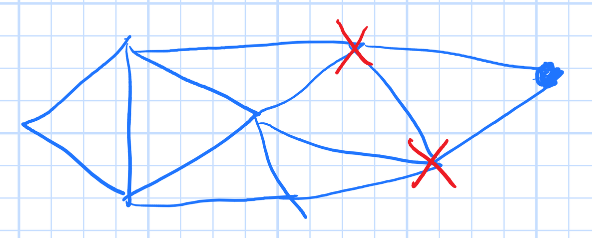
L17: Approximation algorithm

10/19/23

Indepent set

G: Graph
$X \subseteq V(G)$ is independent if $\forall x, y \in X$ $xy \notin E(G)$

Q: max size independent set in G.



$$Q: d_{avg}(G) = \frac{2|E|}{n} = \sum_{v \in V(G)} d(v) / n$$

$$U = \{v \in V \mid d(v) > 2 d_{avg}(G)\}$$

$$|U| \le \frac{n}{2}$$

markov

$$P[d(\text{random vertex}) > 2d_{avg}] \le \frac{E[d(\text{random vertex})]}{2d_{avg}} \le \frac{1}{2}$$

$$W = \{v \in V(G) \mid d(v) \le 2d_{avg}\}$$

$$|W| \ge \frac{1}{2} n$$

while $W \ne \emptyset$ do
  pick $x \in W$
  output x to IS
  remove x and its neighbors from the graph

$$|IS| \ge \frac{|W|}{2d_{avg}+1} \ge \frac{n}{2(2d_{avg}+1)}$$

$$\ge \frac{n}{d_{avg}+1}$$

Turan's theorem

For every graph G with n vertices there is an independent set of size $\ge \frac{n}{d_{avg}+1}$.

Proof Randomly permut the vertices of G $v_1, v_2, \dots, v_n$ be this permutation. Scan the permutation and add a vertex v_i to the IS if you can.
$\Leftrightarrow$ v_i was not deleted
$\Rightarrow$ If we add v_i, we delete all its neighbors.

$$E[|IS|] = E\left[\sum_{v \in V} X_v\right] = \sum_{v \in V} E[X_v] = \sum_{v \in V} \frac{1}{d(v)+1} \ge \frac{n}{\frac{\sum d(v)}{n} + 1}$$

$$= \frac{n}{d_{avg}+1}$$

v — $d(v)$ neighbors

$X_v = 1 \Leftrightarrow v \in IS$ computed

$$P[X_v = 1] \overset{?}{=} \frac{1}{d(v)+1}$$

fact $x, y > 0$

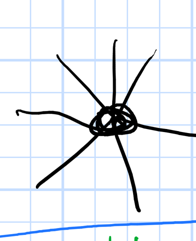
$$\frac{1}{x+1} + \frac{1}{y+1} \ge \frac{2}{\frac{x+y}{2}+1}$$

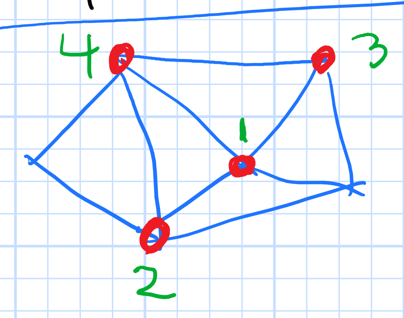
$$\sum_{i=1}^{n} \frac{1}{\alpha_i + 1} \ge \frac{n}{\frac{\sum \alpha_i}{n} + 1}$$

Vertex Cover

$G = (V, E)$

$S \subseteq V$ is a VC if $\forall uv \in E(G)$ $|\{u, v\} \cap S| \ge 1$

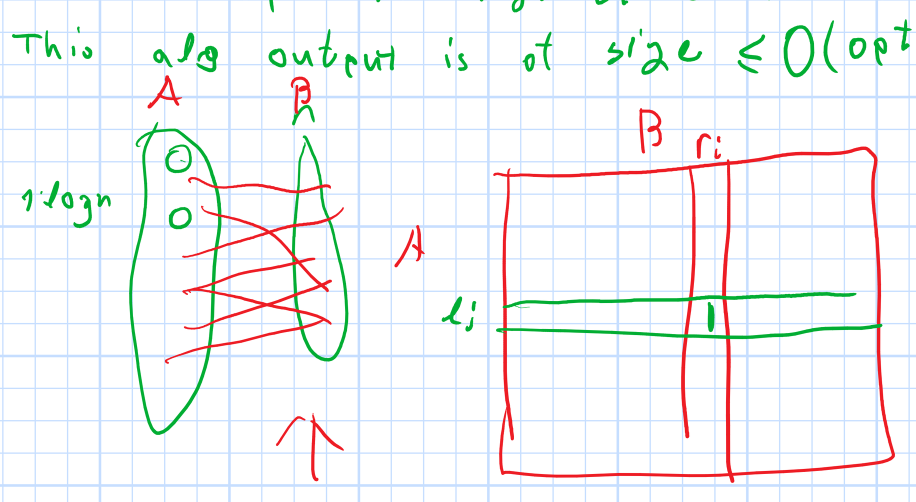


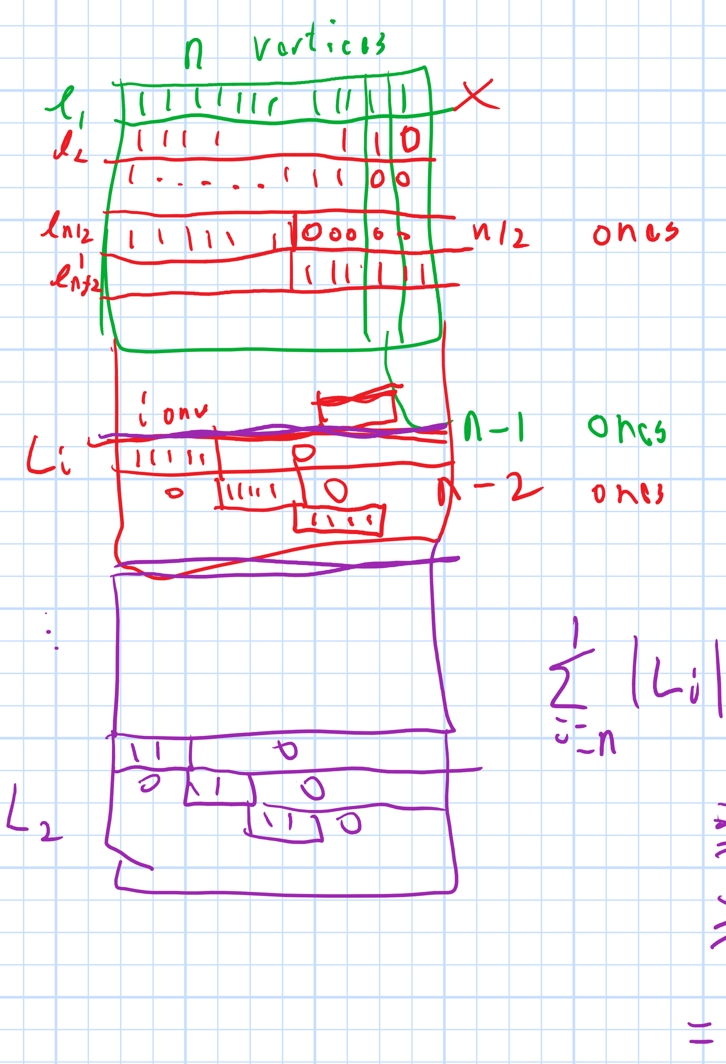


Theorem The greedy alg picking vertex of highest degree in remaining graph to the VC (and deleting all adj edges) outputs a $\Theta(\log n)$ approximation.

opt ~ min size of a VC in G.

This alg output is of size $\le O(\text{opt} \log n)$.





$$\sum_{i=n}^{1} |L_i| = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor$$

$$\ge \sum_{i=1}^{n} \left(\frac{n}{i} - 1\right)$$

$$\ge n \sum_{i=1}^{n} \frac{1}{i} - n$$

$$= \Theta(n \log n - n)$$

$$= \Theta(n \log n).$$

1 for the price of 2

while $uv \in G$ do
  add u, v to VC
  remove u, v and their neighbors from G.

Claim This is a 2-approximation

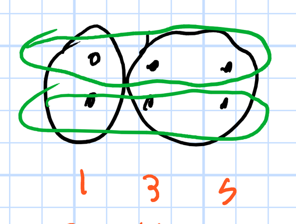
proof

	aprox	opt
first round	u_1 v_1	u_1
	u_2 v_2	v_2
	$\vdots$	
	u_k v_k	$\vdots$

$$|\text{approx sol}| \le 2|\text{opt}|$$

Set Cover

(U, F) — U: set of elements; F: family of subsets of U, $F = \{f_1, f_2, \dots, f_m\}$, $f_i \subseteq U$



$$U = \{1,2,3,4,5,6\}$$

$$F = \{\{1,2\}, \{3,4,5,6\}, \{1,3,5\}, \{2,4,6\}\}$$

Q: Compute min # of sets of F that their union covers all the elements of U

G VC $\Longrightarrow$ Set cover

$$(E(G), \{f_v \mid v \in V(G)\})$$

$$f_v = \{uv \in E(G)\}$$

Approximation alg?

Repeatedly pick set covering largest # of covered elements.

Thm The above algorithm for (U, F) $n = |U|$, output a cover of size $O(k \log n)$

k = min size set cover.

Proof

$O_1, \dots, O_k$: opt. solution

n_i := # elements not covered in begin of ith iteration

$$n_{i+1} = n_i - \#\text{ elements newly covered}$$

$$\le n_i - \frac{n_i}{k}$$

$$\le n_i\left(1 - \frac{1}{k}\right) \le \dots \le \left(1 - \frac{1}{k}\right)^i n$$

$$\le \exp\left(-\frac{i}{k}\right) n < 1$$

$$i = O(k \log n)$$