L13: Min Cut

10/6/23

$\mathcal{E}_i \equiv$ Event 'ith iteration succeeded

The algorithm succeeds if $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \dots \cap \mathcal{E}_n$

$P[\mathcal{E}_i \mid \mathcal{E}_1 \cap \mathcal{E}_2 \cap \dots \cap \mathcal{E}_{i-1}] \equiv$ know value

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$P[A|B] = \frac{P[A \cap B]}{P[B]}$ $\qquad B = \mathcal{E}_1 \quad A = \mathcal{E}_2$

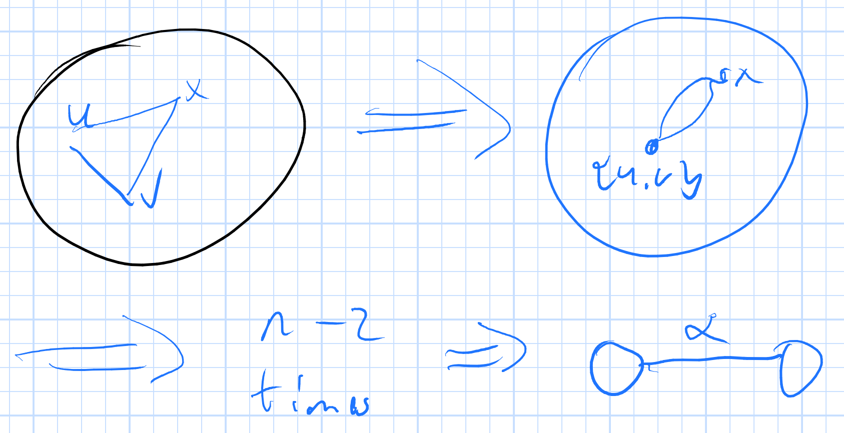
$\Rightarrow P[\mathcal{E}_1 \cap \mathcal{E}_2] = P[\mathcal{E}_1] P[\mathcal{E}_2 | \mathcal{E}_1]$

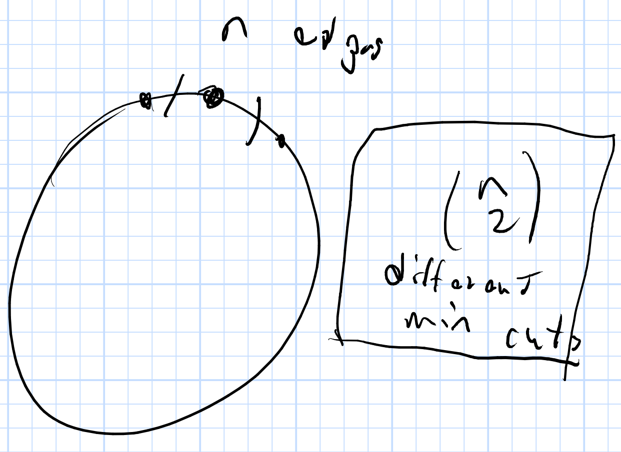
$P[\underbrace{\mathcal{E}_3}_{A} \cap \underbrace{\mathcal{E}_2 \cap \mathcal{E}_1}_{B}] = P[\mathcal{E}_1 \cap \mathcal{E}_2] P[\mathcal{E}_3 | \mathcal{E}_1 \cap \mathcal{E}_2]$

$= P[\mathcal{E}_3 | \mathcal{E}_1 \cap \mathcal{E}_2] P[\mathcal{E}_2 | \mathcal{E}_1] P[\mathcal{E}_1]$

$P[\bigcap_{i=1}^{n} \mathcal{E}_i] = \prod_{i=1}^{n} P[\mathcal{E}_i | \mathcal{E}_1 \cap \dots \cap \mathcal{E}_{i-1}]$

Back to min-cut





$\mathcal{E}_0 \equiv$ prob first contracted edge not in min cut!

$P[\mathcal{E}_0] = ?$

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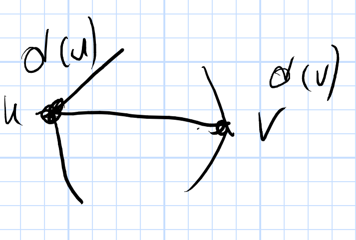
k = size of min cut

$\forall v \quad d(v) \geq k$

$\forall v \in V \quad d(v) \geq k$

$\sum_{v \in V} d(v) = 2m \qquad m = |E(G)|$



$m = \frac{1}{2} \sum_{v \in V} d(v) \geq \frac{1}{2} \sum_{v \in V} k = \frac{nk}{2}$

$P[\mathcal{E}_0] = P\left[\text{picked an edge not in the min cut}\right] = \frac{m-k}{m}$

$= 1 - \frac{k}{m} \geq 1 - \frac{k}{nk/2} = 1 - \frac{2}{n}$,

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$P[\mathcal{E}_1 | \mathcal{E}_0] \geq 1 - \frac{2}{n-1}$

the graph has $n-1$ vertices.

$P[\mathcal{E}_i | \mathcal{E}_0 \cap \mathcal{E}_1 \cap \dots \mathcal{E}_{i-1}] \geq 1 - \frac{2}{n-i}$

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first iter	$\geq 1 - \frac{2}{n}$
2 iter	$\geq 1 - \frac{2}{n-1}$
i	$\geq 1 - \frac{2}{n-i}$
$n-4$	$\geq 1 - \frac{2}{4} = \frac{1}{2}$
$n-3$ iter	$1 - \frac{2}{n-(n-3)} = 1 - \frac{2}{3} = \frac{1}{3}$

$\alpha = P\left[\text{algorithm succeeds}\right] = P[\mathcal{E}_0 \cap \mathcal{E}_1 \cap \dots \mathcal{E}_{n-3}]$

$= \prod_{i=0}^{n-3} P[\mathcal{E}_i | \mathcal{E}_1 \cap \mathcal{E}_2 \cap \dots \cap \mathcal{E}_{i-1}]$

$\geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right) = \prod_i \frac{n-i-2}{n-i}$

$\alpha = \frac{\cancel{n-2}}{n} \frac{\cancel{n-3}}{n-1} \frac{n-4}{\cancel{n-2}} \frac{n-5}{\cancel{n-3}} \cdots \frac{2}{4} \cdot \frac{1}{3}$

$= \frac{2 \cdot 1}{n(n-1)} = \frac{2}{n(n-1)}$

$\left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right) \cdots \frac{1}{3}$

$\alpha \geq \frac{2}{n(n-1)}$ probability of min_cut to output the real min cut.

Run it T times (amplification) $\quad O(n^2)$

Return smallest cut computed

$O(n^2 T)$

Probability that amplified algorithm fail:

$\left(P\left[\text{min-cut failed}\right]\right)^T = (1-\alpha)^T \leq \exp(-\alpha)^T \leq \exp(-\alpha T) \leq \frac{1}{n^3}$

$\leq 1 - \alpha \qquad \boxed{1 - x \leq \exp(-x)} \qquad e^x = \exp(x) \qquad \boxed{T = \frac{1}{\alpha} 3 \ln n}$

$\alpha \geq \frac{2}{n(n-1)}$

$T = \frac{1}{\alpha} 3 \ln n \leq \frac{n^2}{2} \cdot 3 \ln n = O(n^2 \log n)$ times

$O(n^2)$ every run

Total running time is $O(n^4 \log n)$

Implementing contraction

G: adjacency list representation

$\boxed{v} \to \boxed{vu_1} \to \boxed{vu_2} \cdots \boxed{}$

$\boxed{v}$ — $\boxed{\text{list of neighbors}}$ $\qquad 1, 2, \dots, n$

list is sorted

$\boxed{\text{radix sort}}$

$u, v \Rightarrow (u,v)$

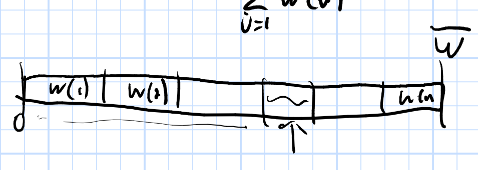
$O(n)$ time merge the lists.

Maintain for every vertex the total weight of edges adjacent to it.

$1, 2, \dots n$

$w(i) > 0$ integer number

Q: How to pick i with probability $\frac{w(i)}{\sum_{j=1}^{n} w(j)} =$



Running time of basic min-cut $O(n^2)$

Shrink(G, t):

Repeatedly do random contraction on G till it has t vertices

return G.

$P\left[\text{Shrink}(G,t) \text{ has same min cut as } G\right] \quad O(n^2)$

$\geq \frac{n-2}{n} \frac{n-3}{n-1} \cdots \frac{t}{t+2} \frac{t-1}{t+1} = \frac{t(t-1)}{n(n-1)} \geq \frac{1}{2}$

$t = \frac{n}{\sqrt{2}}$

FastCut(G)

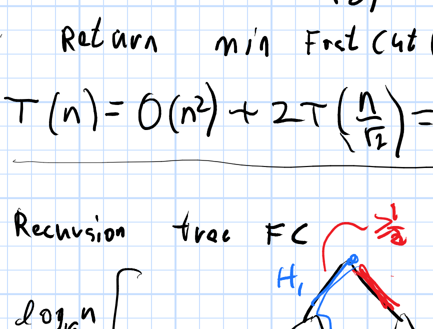
$H_1 \leftarrow \text{Shrink}(G, \frac{n}{\sqrt{2}})$ (copy)

$H_2 \leftarrow \text{Shrink}(G, \frac{n}{\sqrt{2}})$

Return min FastCut(H_1), FC(H_2))

$T(n) = O(n^2) + 2T\left(\frac{n}{\sqrt{2}}\right) = O(n^2 \log n)$

Recursion tree FC



$P\left[\exists \text{ blue path root to leaf}\right] \geq \frac{1}{h+1} = \frac{1}{\log_{\sqrt{2}} n + 1} = \Omega\left(\frac{1}{\log n}\right)$

$\alpha = \frac{1}{\log n}$ prob FC succeed

$T = \frac{1}{\alpha} 3 \ln n = O(\log^2 n)$

$(1-\alpha)^T \leq \exp(-\alpha T) \leq \frac{1}{n^3}$

$O(n^2 \log n) \cdot O(\log^2 n)$

$= O(n^2 \log^3 n)$!!!