473 L12 : More randomization 10/3/23

Chernoff's inequality

$X_1, X_2, \dots, X_m \in \{0,1\}$ independent coin flips.

$Y = \sum X_i$ $\quad E[Y] = \frac{m}{2}$

$P[Y < \frac{m}{4}] = P[Y \le (1-\frac{1}{2})E[Y]] \le \exp(-\frac{m}{c})$

c is some constant.

$X_1, X_2, \dots, X_n \in \{-1,+1\}$ independently, $P[X_i = 1] = \frac{1}{2}$

$Y = \sum_{i=1}^{n} X_i$

Linearity of E

$E[Y] \overset{!}{=} E[\sum X_i] = \sum_i E[X_i] = \sum 0 = 0.$

$E[Y^2] = E((\sum X_i)^2] = E[(X_1 + X_2 + \dots + X_n)^2]$

$= E[X_1^2 + X_2^2 + \dots + X_n^2 + 2\sum_{i<j} X_i X_j]$

$= \sum_{i=1}^{n} E[X_i^2] + 2\sum_{i<j} E[X_i X_j]$

$= n + 0 = n$

$E[X_i^2] = 1$

$E[X_i X_j] = E[X_i]E[X_j] = 0$

$P[|Y| \ge t\sqrt{n}] = P[Y^2 \ge t^2 n] = P[Y^2 \ge t E[Y^2]] \le \frac{1}{t^2}$

Markov's inequality

$V[Y] = E[(Y - E[Y])^2]$

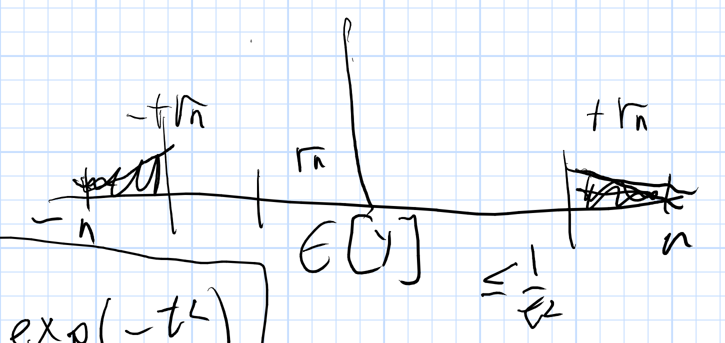
$= E[Y^2] - (E[Y])^2$ variance of Y

Chebychev inequality

For any random variable Y

$P[|Y - E[Y]| \ge t\sigma_Y] \le \frac{1}{t^2}$

$\sigma_Y = \sqrt{V[Y]}$ standard deviation.



$\boxed{P[Y \text{ in tail}] \le \exp(-\frac{t^2}{c})}$ Chernoff.

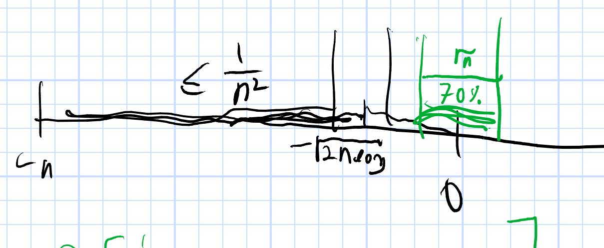
$n = 1000000$

$\sqrt{n} = 1000$

$500{,}000 \pm 2\sqrt{1000}$

$Y \in [-n, n]$

$\boxed{P[|Y| > t\sqrt{n}] \le 2\exp(-\frac{t^2}{4})}$



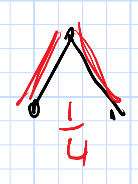
$P[\text{in } n \text{ coin flips you got } i \text{ heads}] = \frac{\binom{n}{i}}{2^n}$

T_h full binary tree of height n

randomly color the edges of T_h by red and blue. Q: what is the probability of a blue path from the root to any leaf.

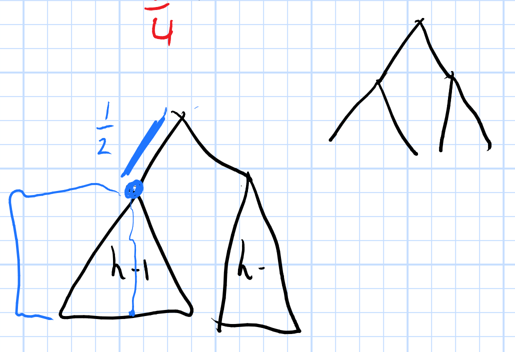
P_n

$P_0 = 1 \quad P_1 = 1 - \frac{1}{4} = \frac{3}{4}$



$P_2 =$

$P_n =$



left $\frac{1}{2}P_{n-1}$

right

success $\frac{1}{2}P_{h-1}$

$\boxed{P_n = 1 - (1 - \frac{P_{n-1}}{2})(1 - \frac{P_{n-1}}{2})}$

$P_n = 1 - (1 - P_{n-1} + \frac{P_{n-1}^2}{4}) = P_{n-1} - \frac{P_{n-1}^2}{4}$

$P_1 = \frac{3}{4}$

$P_2 = \frac{3}{4} - \frac{1}{4}(\frac{3}{4})^2 = \frac{48 - 9}{64} = \frac{39}{64}$

Lemma $P_n \ge \frac{1}{n+1}$ $\quad \forall n$ (see class notes for proof)

$P_n = O(\frac{1}{n+1})$

Min Cut

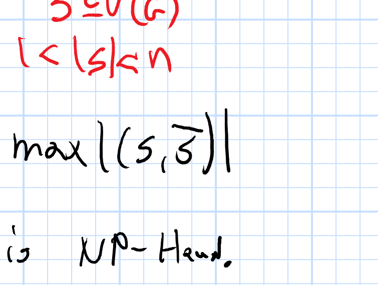
G: graph undirected n vertices

m edges.

def The min-cut in G is the minimal # of edges that one has to remove to make G disconnected.

Cut

$S \subseteq V(G)$ $\quad \bar{S} = V \setminus S$

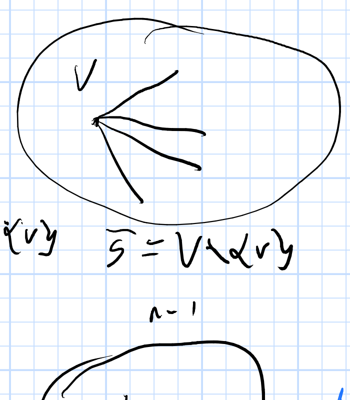


$(S, V\setminus S) = (S, \bar{S}) = \{e \in E(G) \mid e = uv, |\{u,v\} \cap S| = 1\}$

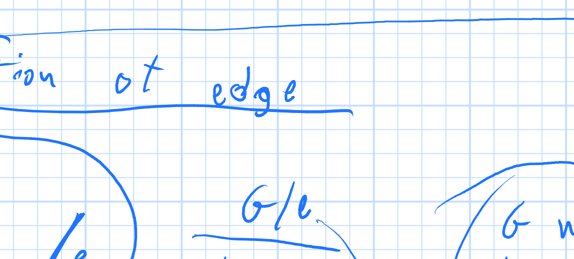
$\text{MinCut}(G) = \arg\min_{\substack{S \subseteq V(G) \\ 1 < |S| < n}} |(S, \bar{S})|$

$\text{MaxCut}(G) = \arg\max |(S, \bar{S})|$

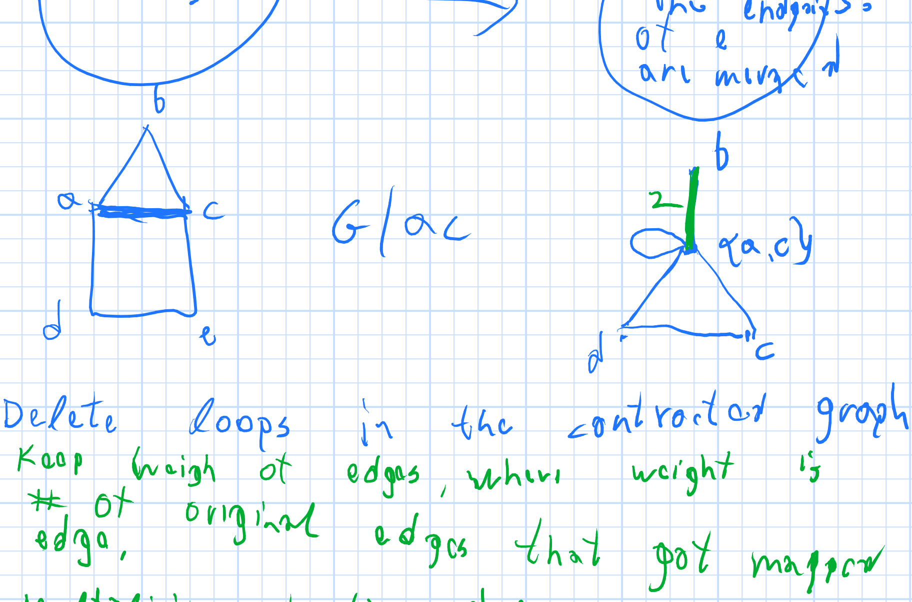
Thm MaxCut is NP-Hard.



$S = \{v\}$ $\quad \bar{S} = V \setminus \{v\}$ $\quad (S, \bar{S}) =$ all edges adjacent to v



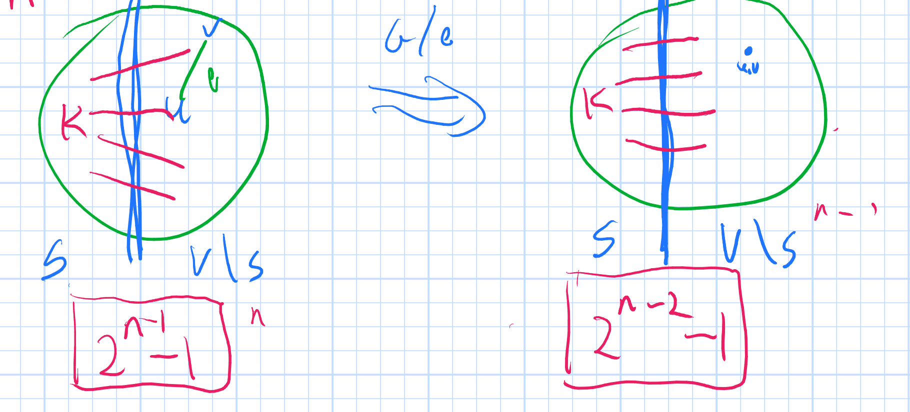
Contraction of edge

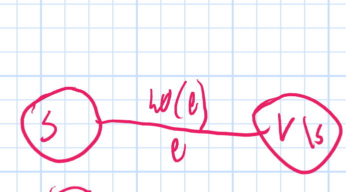


Delete loops in the contracted graph

- Keep weight of edges, where weight is # of original edges that got mapped to this edge.

Multiplicity of the edge.





$\alpha := P[\text{contract randomly alg outputs min cut}] \ge \frac{1}{n^2}$

$O(\frac{\log n}{p}) = O(n^2 \log n)$ times.