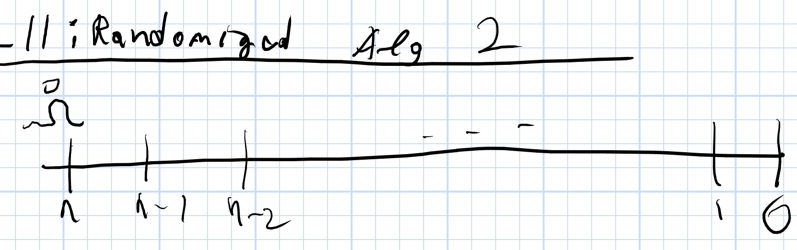
L11: Randomized Alg 2 | 9/26/23



If frog at location t then it jumps to location αt. $\alpha \in (0,1)$ $\alpha = 7/8$

$t \to \lfloor \alpha t \rfloor$

$\lfloor \alpha \lfloor \alpha n \rfloor \rfloor \le \alpha^2 n$

$Y_i \equiv$ location at time i

$Y_0 = n$

$Y_i = \lfloor \alpha Y_{i-1} \rfloor \le \alpha Y_{i-1} \le \alpha^i n.$

Q: first i s.t. $Y_i < 1$

$\equiv$ # of rounds till frog arrives to 0.

$\alpha^i n < 1 \iff \alpha^i < \frac{1}{n} \iff n < \frac{1}{\alpha^i}$

$\iff \log n < i \log \frac{1}{\alpha}$ $\qquad \alpha = \frac{7}{8}$

$i = \left\lceil \frac{\log n}{\log 1/\alpha} \right\rceil = \lceil \log_{1/\alpha} n \rceil = O(\log n)$

Randomized frog

If at location t it jumps to location X (random variable)

$E[X] \le \alpha t$

$Y_0 = n$, Y_i = location of frog after i rounds

$E[Y_1] \le \alpha Y_0 = \alpha n$

$E[Y_2]$

$E[Y_2 \mid Y_1 = y] \le \alpha y$

$E[Y_2] = E\left[E[Y_2 \mid Y_1]\right] = E_y\left[E[Y_2 \mid Y_1 = y]\right]$

$\le \underset{y \sim Y_1}{E}[\alpha y] = \alpha E[Y_1] \le \alpha \cdot \alpha n$

$= \alpha^2 n.$

$E[Y_i \mid Y_{i-1}] \le \alpha Y_{i-1}$

$E[Y_i] = E\left[E[Y_i \mid Y_{i-1}]\right] \le E[\alpha Y_{i-1}] = \alpha E[Y_{i-1}]$

$\le \alpha \cdot \alpha^{i-1} n$

$= \alpha^i n.$

Probability that the frog plays for m rounds and yet did not arrive to 0.

$P[Y_m > 0] = P[Y_m \ge 1] \le \frac{E[Y_m]}{1} = E[Y_m] \le \alpha^m n$

$\boxed{m = 4 \cdot \log_{1/\alpha} n}$

$P[Y_m \ge 1] \le E[Y_m] \le \alpha^m n = \frac{1}{n^4} \cdot n = \frac{1}{n^3}$

$\alpha^{\log_{1/\alpha} n} = \frac{1}{n}$

Lemma The prob that the random frog would not arrive to 0, when starting at n, and jumping in expectation to location $\le \alpha$(curr-loc), after $m = 4 \log_{1/\alpha} n$, is at most $\frac{1}{n^3}$.

Theorem

QS when run on input of size n takes $O(n \log n)$ time, with probability $\ge 1 - \frac{1}{n^2}$.

with high probability

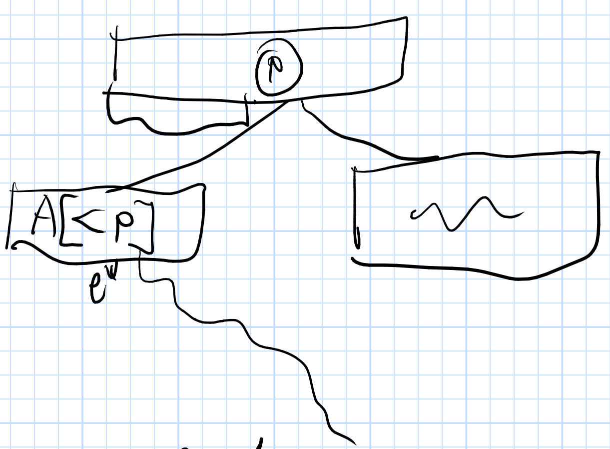
Proof

$A[1 \dots n]$ input.

e specific element in A

$S_i \subseteq A[1, \dots, n]$ the subproblem of QS in the ith level of the recursion that includes e.

S_1

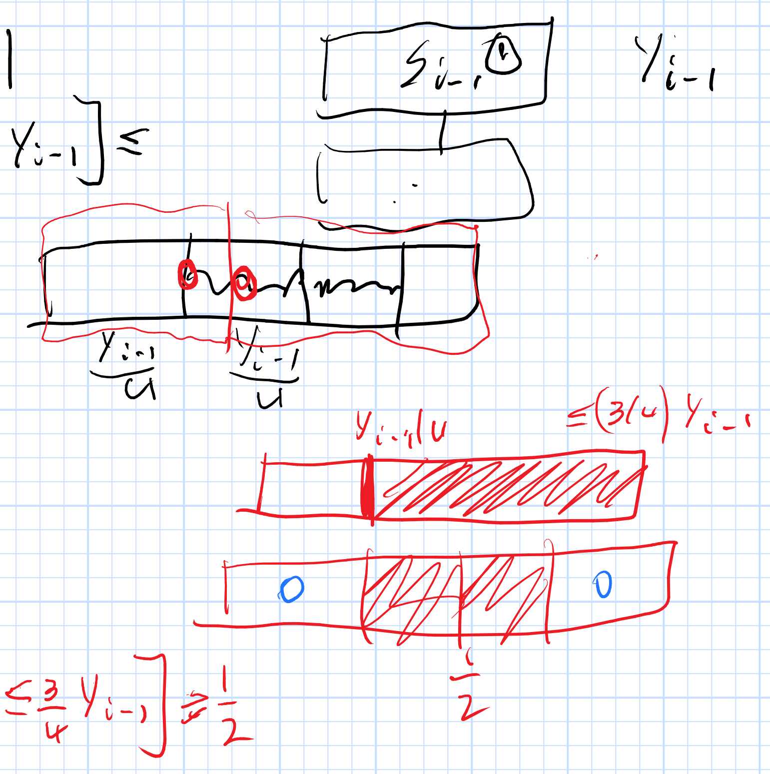


How many levels of the recursion e participates in?

$Y_0 = |S_0|$

$Y_i = |S_i|$

$E[Y_i \mid Y_{i-1}] \le$



$P\left[Y_i \le \frac{3}{4} Y_{i-1}\right] \ge \frac{1}{2}$

$E[Y_i \mid Y_{i-1}] \le \frac{1}{2} \cdot \frac{3}{4} Y_{i-1} + \frac{1}{2} Y_{i-1} = \frac{7}{8} Y_{i-1}$

$\alpha = 7/8$

$P\left[e \text{ to participate in } \boxed{4 \log_{8/7} n}\right] \le \frac{1}{n^3}$ (from the frog lemma)

$\mathcal{E}_i \equiv$ the ith element in the array participates in more than $\underbrace{4 \log_{8/7} n}_{M}$ comparisons

$P[\mathcal{E}_i] \le \frac{1}{n^3}$

$P\left[\text{QS performs more than } \overset{O(n \log n)}{nM} \text{ comparisons}\right]$

$\le P\left[\bigcup_i \mathcal{E}_i\right] \overset{\text{union bound}}{\le} \sum P[\mathcal{E}_i] \le n \cdot \frac{1}{n^3} = \frac{1}{n^2}$ ∎

Treaps

$e_1, e_2, \dots, e_n$ elements

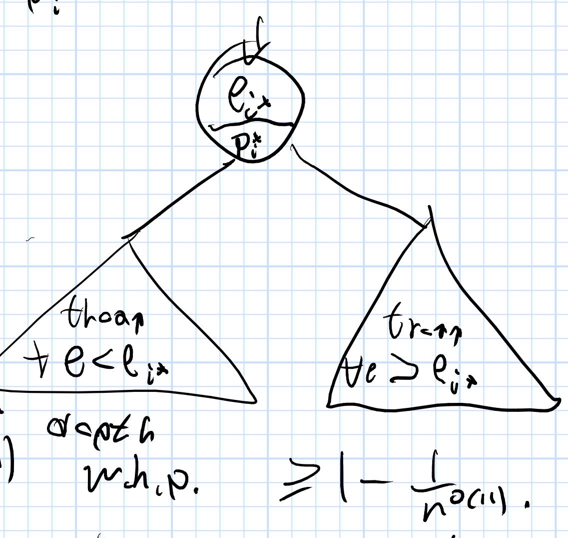
Generate random priorities

$p_i \in [0,1]$ $i = 1, \dots, n$

$p_i \equiv$ random priority for element.

Building the treap

$i^* = \arg\min_i p_i$

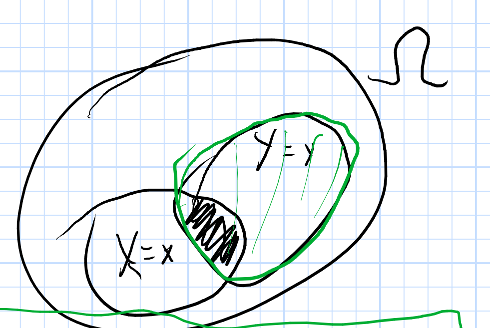


Claim Treap depth is $O(\log n)$ w.h.p. $\ge 1 - \frac{1}{n^{O(1)}}$.

proof Treap $\equiv$ recursion tree of QS. Done. ∎

Conditional probability/expectation

$P[X = x \mid Y = y]$ = probability of X to be x given that $Y = y$



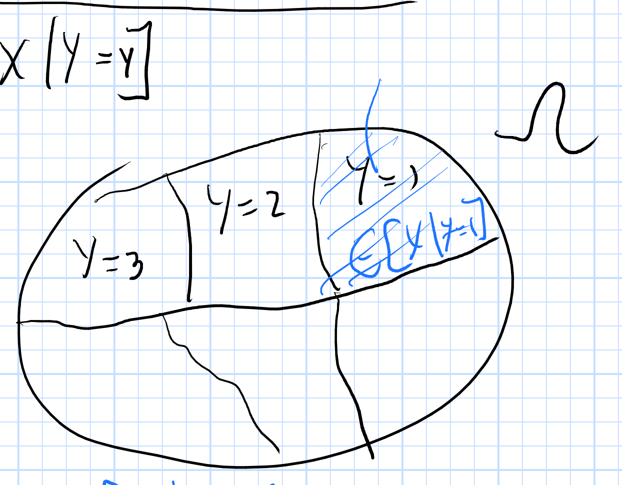
$\boxed{P[X = x \mid Y = y] = \frac{P[X = x \cap Y = y]}{P[Y = y]}}$ Conditional probability

$P[X = x \mid Y = y] : [\text{values of } y] \longrightarrow$ probability

$E[X \mid Y = y]$ = Expected value of X given $Y = y$

$= \sum_x x P[X = x \mid Y = y]$

$f(y) = E[X \mid Y = y]$



$E[X \mid Y] = E[X \mid Y = y]$

$E\left[E[X \mid Y]\right] = E_y\left[E[X \mid Y = y]\right]$

$= \sum_y P[Y = y] E[X \mid Y = y]$

$\boxed{E\left[E[X \mid Y]\right] = E[X]}$

Markov's Inequality

X: Random variable

$X \ge 0$ (only non-negative values)

$P[X \ge t] \le \frac{E[X]}{t}$

Proof

$E[X] = \sum_{x \ge 0} x P[X = x]$

$= \sum_{x < t} x P[X = x] + \sum_{x \ge t} x P[X = x]$

$\ge 0 + t \sum_{x \ge t} P[X = x] = t P[X \ge t]$

$P[X \ge t] \le \frac{E[X]}{t}$. ∎

Union bound

$\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n$ n events

$P\left[\bigcup_{i=1}^n \mathcal{E}_i\right] \le \sum_{i=1}^n P[\mathcal{E}_i]$.