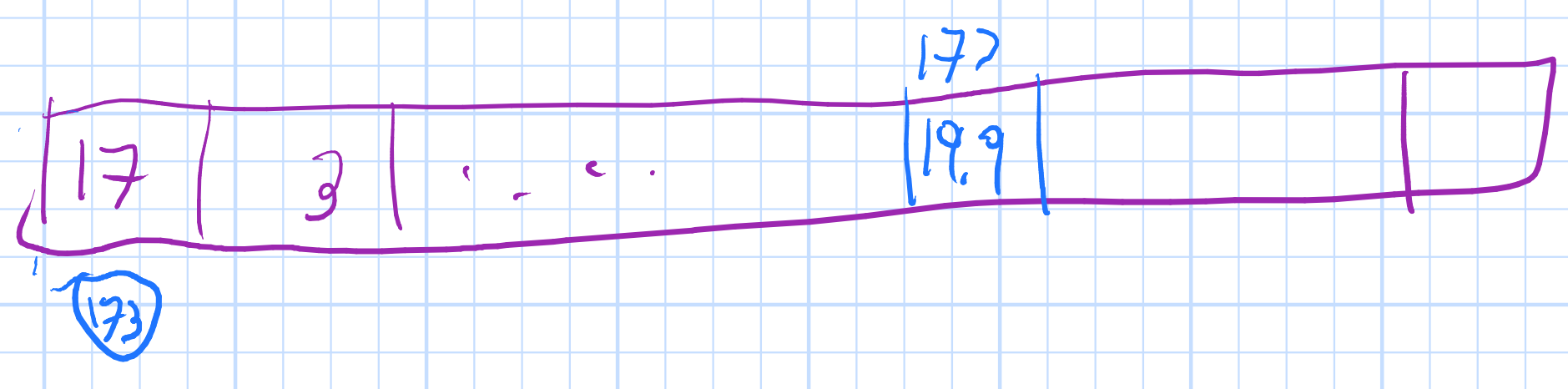
L10: Randomized Algorithms

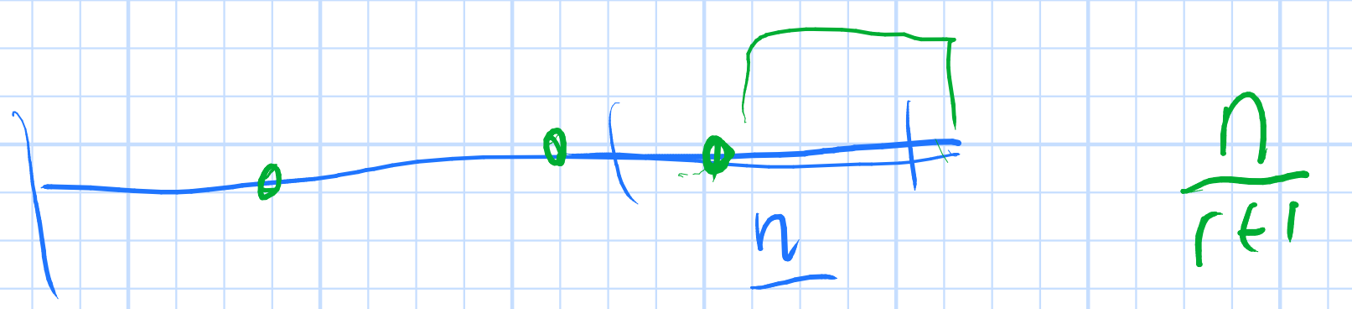
9/21/23



Q: Compute the max value in the array?

r values $v_1, v_2, \dots, v_r$

arg max$(v_1, \dots, v_r)$

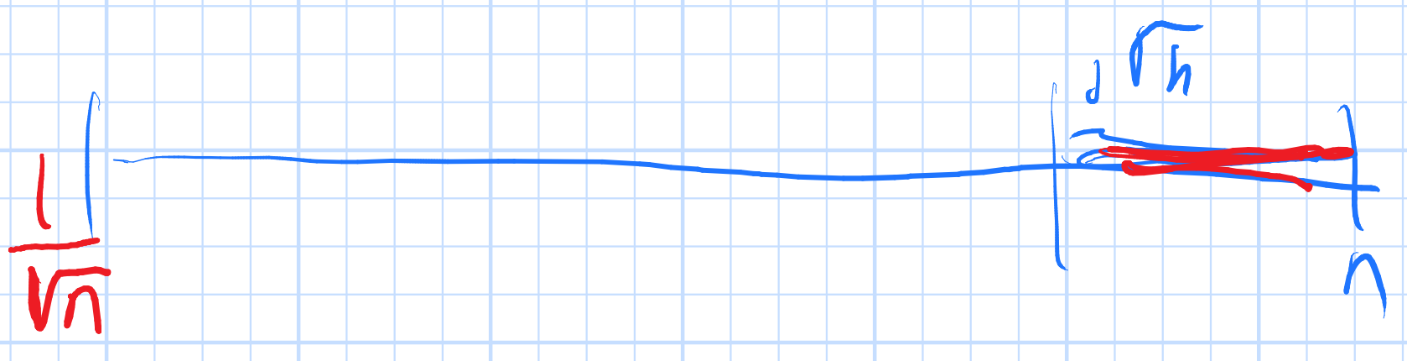


$O(r + Y_r)$

↑ lenght at head for a max element encountered

Follow and improve

```
s ← 1
while next(s) ≠ null do
    s ← next(s)
    r ← pick random value in array
    s ← max(s, r)
return s.
```



$p = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

$I \equiv$ first iteration with value in last $\sqrt{n}$ locations.

$O(I + \sqrt{n})$

$E[I] = \frac{1}{p} = \sqrt{n}$

$X \sim Geo(p)$

if $P[X = i] = p^i$

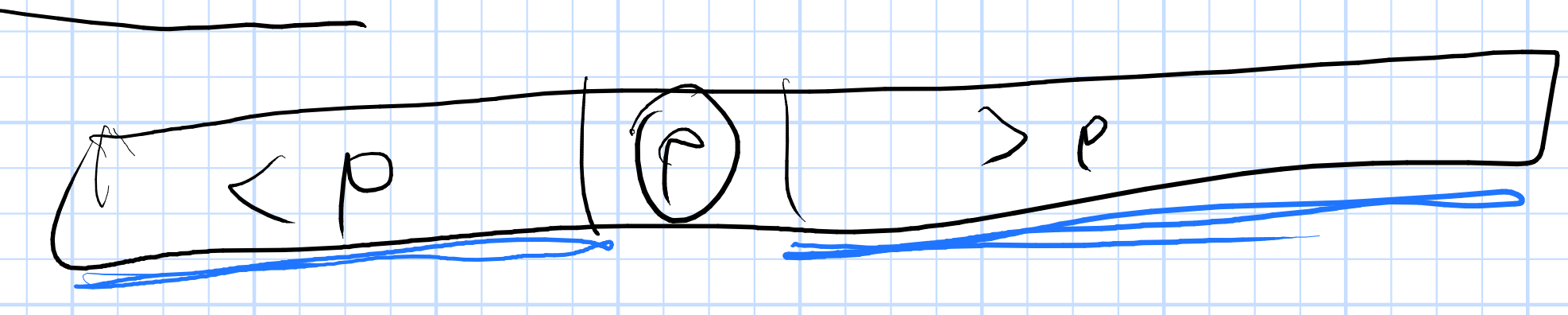
$E[X] = \sum_{i=1}^{\infty} i \cdot P[X = i] = \sum_{i=1}^{\infty} i p^i = \frac{1}{p}$.

Lemma Previous alg has expected running time $O(\sqrt{n})$.

$O(n^{c/2})$

Nuts and Bolts

Quick Sort



Indicator variable

$X = 0$ or 1

$\mathcal{E}$: some event

X is an indicator variable for $\mathcal{E}$

$\iff X = 1$ iff $\mathcal{E}$ happened

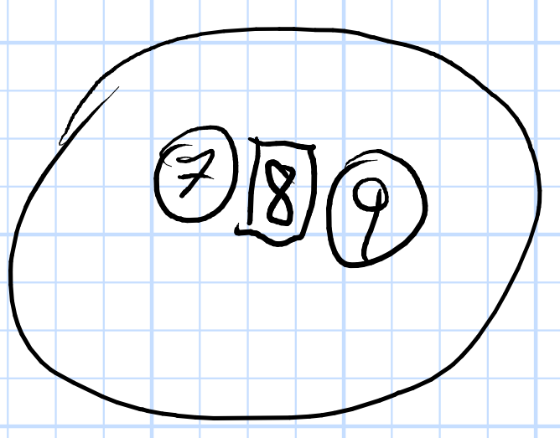
$X_{i,j} = 1 \iff$ the ith rank element in the input got compared to the jth rank element.

The rank of an element is its location in the sorted array

$X_{i,j} = 1$

$P[X_{i,i+1} = 1] = 1$

$P[X_{i,i+2} = 1] = \cancel{1/2}\ \frac{2}{3}$

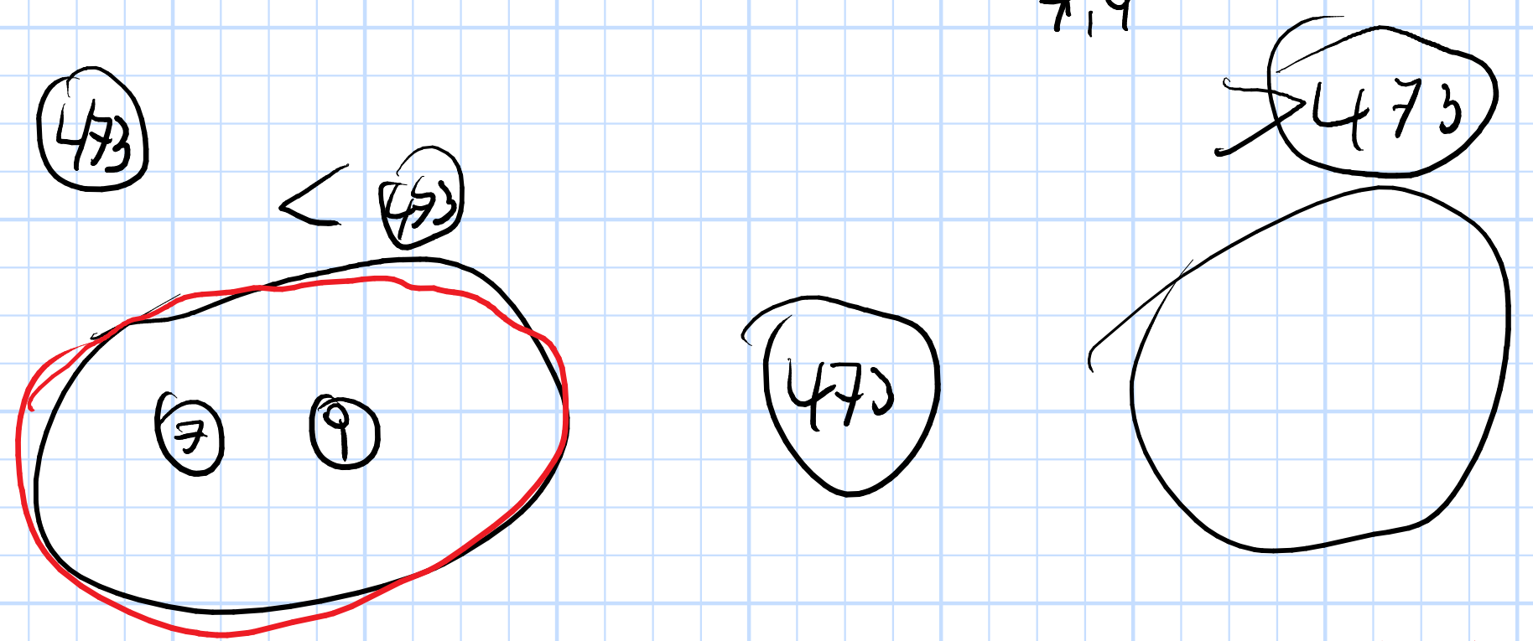


⑦ and ⑨ saprate if ⑧ is the pivot.

If ⑦ is the pivot. then $X_{7,9} = 1$

If ⑨ is the pivot then $X_{7,9} = 1$

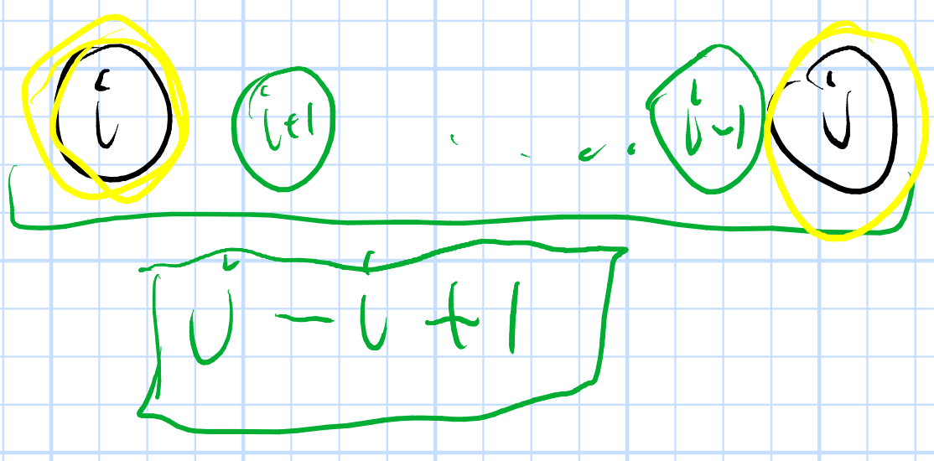
If (473)



$P[X_{7,9} = 1] = P[⑦ \text{ or } ⑨ \text{ are the pivot} \mid \text{the pivot is} \in \{⑦, ⑧, ⑨\}] = \frac{2}{3}$

$i < j$

$P[X_{i,j} = 1] = \frac{2}{j - i + 1}$



$\frac{2}{j - i + 1}$

Total # of comp permormed by QS is

$Y = \sum_{i<j} X_{i,j}$

Linearity of expectations

$\forall$ R.V. X and Y we have

$E[X + Y] = E[X] + E[Y]$

$E[Y] = E\left[\sum_{i<j} X_{i,j}\right] \underset{\uparrow}{=} \sum_{i<j} E[X_{i,j}]$

linearity of expectations

$= \sum_{i<j} P[X_{i,j} = 1] = \sum_{i<j} \frac{2}{j - i + 1} \underset{\uparrow}{\leq} \sum_{i=1}^{n} \sum_{j=i+1}^{i+n-1} \frac{2}{i - j + 1}$

$\alpha = \sum_{j=i+1}^{i+n-1} \frac{2}{j - i + 1} = \sum_{\ell=2}^{n} \frac{2}{\ell} \leq 2 H_n \leq 2(\ln n + 1)$

H_n ↗ harmonic number

$E[Y] \leq \sum_{i=1}^{n} \alpha \leq 2(\ln n + 1)\, n = O(n \log n)$

$= \boxed{2n \ln n + 2n}$

$E[X_I] = 0 \cdot P[X_I = 0] + 1 \cdot P[X_I = 1] = P[X_I = 1]$

↑ indicator variable

↑ probability of event being indicated by var