

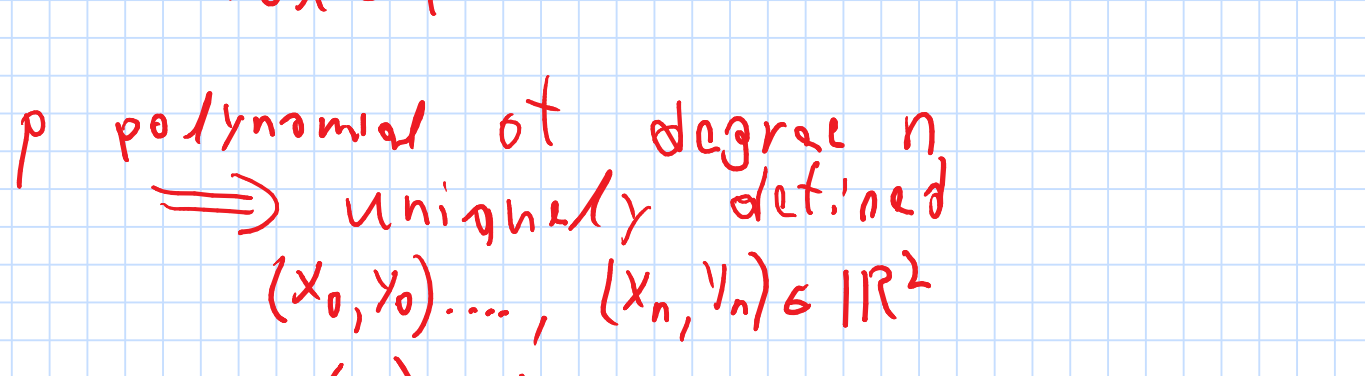
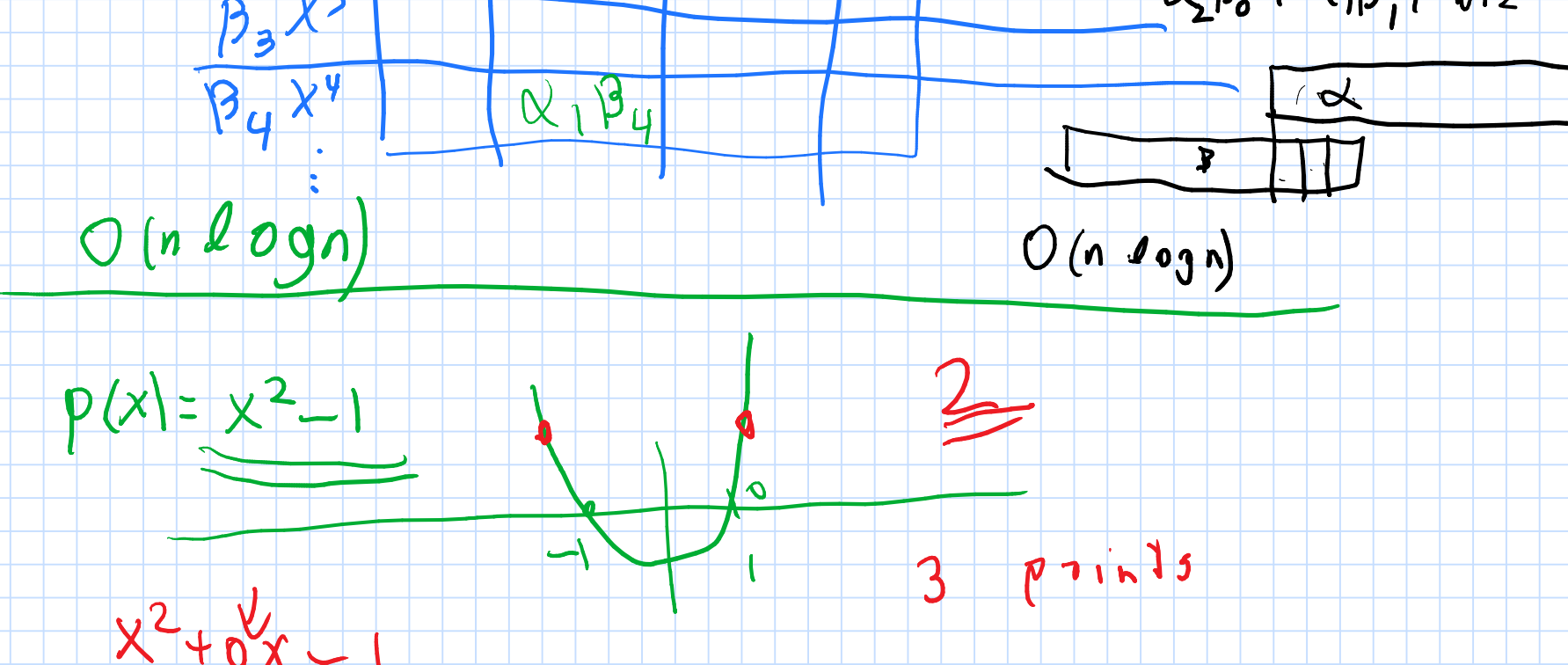
Lecture 2 Fast Fourier Transform

$$p(x) = \sum_{i=0}^{n-1} \alpha_i x^i \quad q(x) = \sum_{i=0}^{n-1} \beta_i x^i \quad n \text{ coefficients } \beta_0, \beta_1, \dots, \beta_{n-1}$$

Q: compute $p(x) \cdot q(x)$?

$$r(x) = p(x)q(x)$$

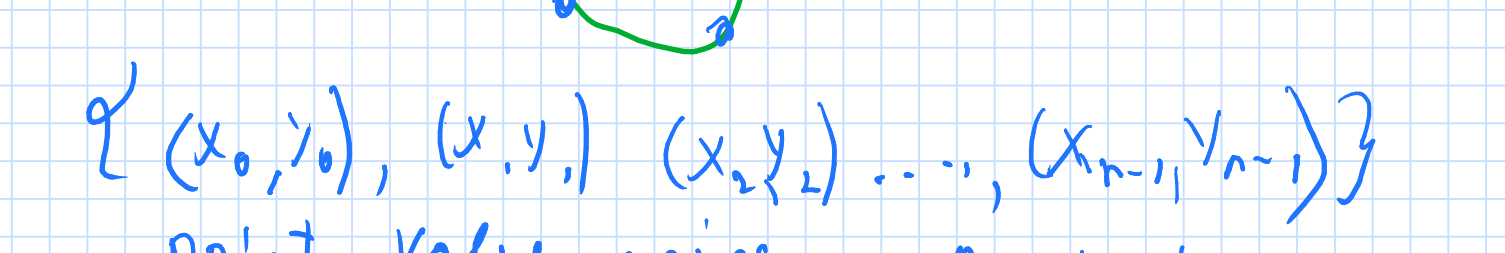
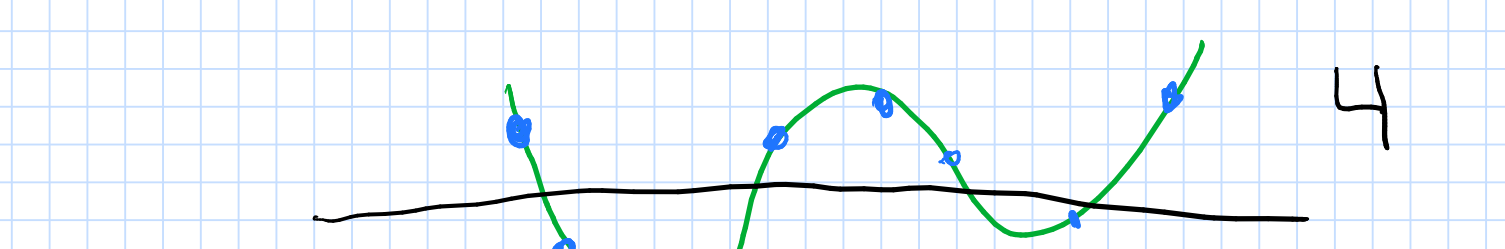
$$r(x) = \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \alpha_i \beta_j x^{i+j} = \sum_{k=0}^{2n-2} \sum_{i=\max(k-(n-1), 0)}^{\min(n-1, k)} \alpha_i \beta_{k-i} x^k$$



p polynomial of degree $n \Rightarrow$ uniquely defined $(x_0, y_0), \dots, (x_n, y_n) \in \mathbb{R}^2$

$$p(x_i) = y_i$$

p is of degree n but it is zero $x_1, \dots, x_n$ different values then it is the zero polynomial $p(x) = 0$



$\{(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_{n-1}, y_{n-1})\}$ point value pairs $n-1$ degree

$$p(x) = (x_i, y_i) \quad i=0, \dots, 2n$$

$$q(x) = (x_i, z_i) \quad i=0, \dots, 2n$$

$$p(x_i) = y_i \quad q(x_i) = z_i \quad \forall i$$

$$r(x) = p(x)q(x) \quad \text{degree}(r) = 2n-2$$

$$= (x_i, p(x_i)q(x_i)) \quad i=0, \dots, 2n$$

$$= (x_i, y_i z_i) \quad i=0, \dots, 2n$$

$O(n)$ time!

$p(x)$ of degree $n-1 \Rightarrow (x_i, y_i) \quad i=0, \dots, n-1$

$(x_i, y_i) \quad i=0, \dots, n-1$ compute $r(x)$ Move back to standard rep?

$$O(n \log n)$$

$$(x_i, y_i) \quad i=0, \dots, n-1 \quad p(x) = \sum_{i=0}^{n-1} y_i \prod_{j \neq i} \frac{(x-x_j)}{(x_i-x_j)}$$

$$\Rightarrow X = \{x_0, \dots, x_{n-1}\} \quad p(x) = \sum \alpha_i x^i$$

$$p(x) = \{(x_0, p(x_0)), (x_1, p(x_1)), \dots, (x_{n-1}, p(x_{n-1}))\}$$

n is a power of 2

$$p(x) = \underbrace{\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots}_{u(x^2)} + \underbrace{\alpha_3 x^3 + \dots}_{xv(x^2)}$$

$$p(x) = u(x^2) + xv(x^2)$$

$$p(x) = u(x^2) + xv(x^2)$$

$$X^2 = \{x^2 \mid x \in X\}$$

$$A_{\text{eg}} \text{ FFT}(p(x) = \sum \alpha_i x^i, X)$$

$$u(x) = \sum_{i=0}^{n/2-1} \alpha_{2i} x^i \quad v(x) = \sum_{i=0}^{n/2-1} \alpha_{2i+1} x^i \quad O(n)$$

$$X^2 = \{x^2 \mid x \in X\}$$

$$u[X^2] \leftarrow \text{FFT}(u, X^2) \quad v[X^2] \leftarrow \text{FFT}(v, X^2)$$

$$\text{For } x \in X \text{ do } \text{out} \leftarrow \text{out} + \{x, u(x^2) + xv(x^2)\}$$

$$\text{return out} \quad O(n)$$

$$X^2$$

$$|X| = n \quad |X^2| \leq n$$

$$X = \{x_0, x_1, \dots, x_{n-1}\}$$

$$X = \{x_0, -x_0, x_1, -x_1, \dots, x_{n/2-1}, -x_{n/2-1}\}$$

$$|X^2| = \frac{n}{2}$$

$$X \rightarrow X^2 \rightarrow X^4$$

$$n \quad \frac{n}{2} \quad \frac{n}{2}$$

$$|X^4| = \frac{n}{4}$$

$$|X^8| = \frac{n}{8}$$

$$|X^{2^i}| = \frac{n}{2^i}$$

$$O(\frac{n}{2^i}) \quad \frac{n}{2^i}, \frac{n}{2^i}$$

$$T(n) = O(n) + 2T(\frac{n}{2}) = O(n \log n)$$

$$\text{roots of unity}$$

$$x + iy$$

$$r = 1$$

$$(r, \alpha) = r \cos \alpha + i r \sin \alpha$$

$$(r', \alpha')$$

$$(rr', \alpha + \alpha')$$

$$1, -1, i, -i$$

$$\omega_k^{(n)} = \cos\left(\frac{2\pi}{n}k\right) + i \sin\left(\frac{2\pi}{n}k\right)$$

$$k=0, \dots, n-1 \quad n \text{ is power of 2}$$

$$X = \{\omega_k^{(n)} \mid k=0, \dots, n-1\}$$

$$X^2 = \{\omega_k^{(n/2)} \mid k=0, \dots, n/2-1\}$$

$$|X^{2^i}| = \frac{n}{2^i}$$

$$p(x) = \sum_{i=0}^{n-1} \alpha_i x^i \quad X = \{x_0, x_1, \dots, x_{n-1}\}$$

$$y_i = p(x_i) \quad i=0, \dots, n-1$$

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$$

$$V \quad X_i = \omega_i^{(n)}$$

$$\text{Vandermonde matrix}$$

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = V \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$$

$$V^{-1}$$

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = V^{-1} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

$$V^{-1} = \frac{1}{n}$$

$$V \downarrow \text{write rows in reverse order}$$

$$V^{-1}$$

$$V^{-1} = \frac{1}{n} \begin{pmatrix} \text{row} \\ \text{reverse} \\ \text{of } V \end{pmatrix}$$

$$V^{-1} \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} \Leftarrow V \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

$$O(n) \quad O(n \log n)$$

$$\omega_k^{(n)}$$

$$\omega_k^{(n)}$$

$$\omega_k^{(n)}$$

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