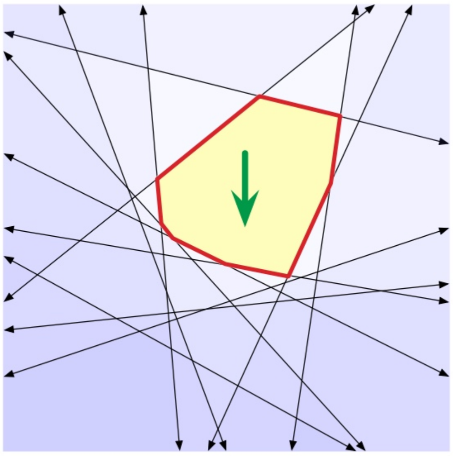
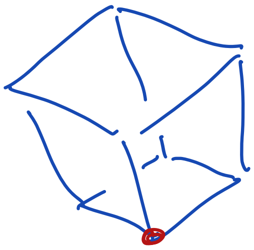
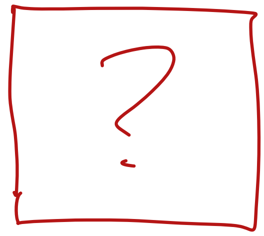
LP = Find lowest point in convex polyhedron



2 variables
n+2 constraints



Dual



n variables
n+2 const.

Primal (Π)	$\Longleftrightarrow$	Dual (∐)
max $c \cdot x$ s.t. $Ax \le b$ $x \ge 0$		min $y \cdot b$ s.t. $yA \ge c$ $y \ge 0$

d variables
n matrix constraints
d sign constraints

n variables
d matrix constraints
n sign constraints

The Fundamental Theorem of Linear Programming. *A canonical linear program Π has an optimal solution x^* if and only if the dual linear program ∐ has an optimal solution y^* such that* $\underbrace{c \cdot x^*}_{} = y^*Ax^* = \underbrace{y^* \cdot b}_{}$.

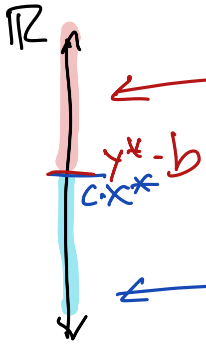
same objective value

Weak Duality: If x is feasible for primal LP
y is feasible for dual LP

Then $c \cdot x \le yAx \le y \cdot b$

Proof: x is feasible $\Rightarrow$ $Ax \le b$
y is feasible $\Rightarrow$ $y \ge 0$ $\Rightarrow$ $yAx \le y \cdot b$

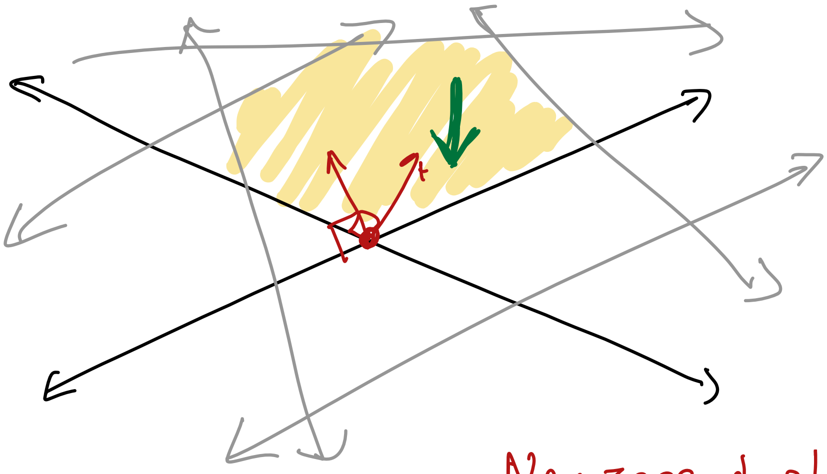
Symmetric $\Rightarrow$ $c \cdot x \le yAx$ $\square$



$\{y \cdot b \mid yA \ge c, y \ge 0\}$

Strong duality: NO GAP

$\{c \cdot x \mid Ax \le b, x \ge 0\}$

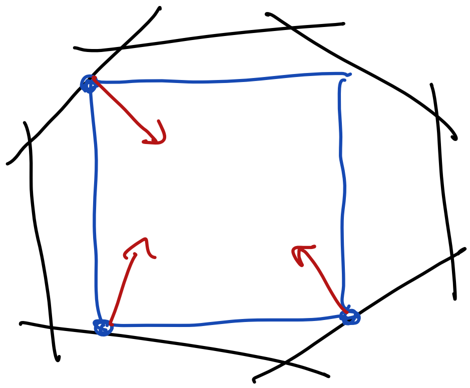


Dual variable for each (matrix) constraint

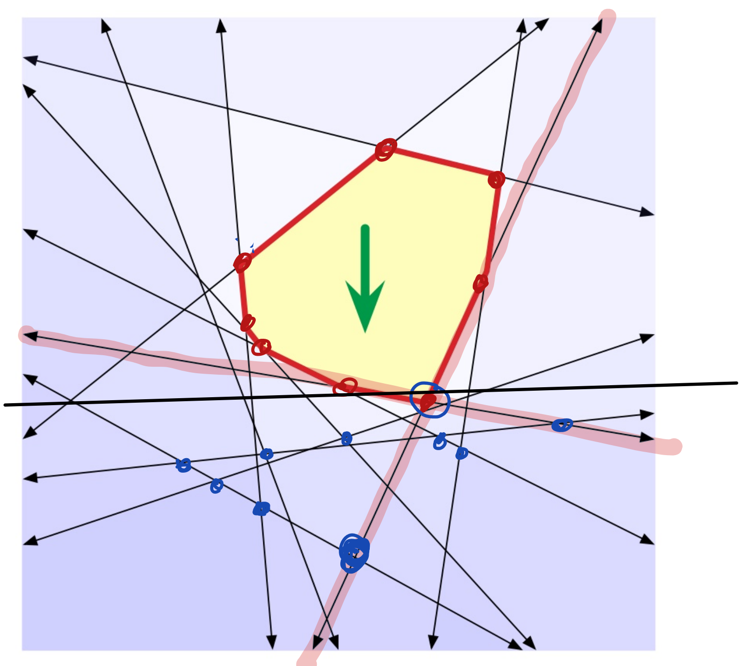
If primal constraint is not tight for x^*

then corr dual variable is 0

Non zero dual variables



coefficients of $-c$ in coord frame defined by normals to tight constraints



basis = d ~~linearly independent~~ constraints

Assume no degeneracies

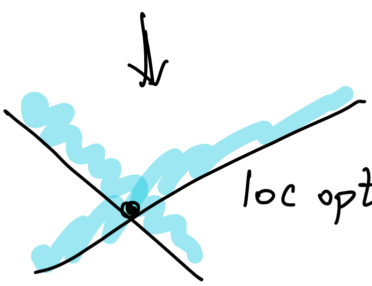
location = solution of $d \times d$ linear system
= intersection of d constraint planes

value = $c \cdot$ location

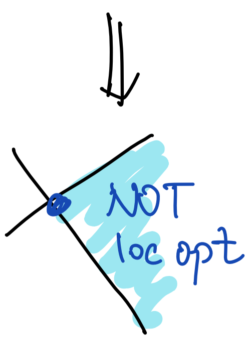
There are $\binom{n+d}{d} = \binom{d+n}{n}$ bases

Basis is feasible if

$Ax \leq b$ where x = location
$x \geq 0$



loc opt

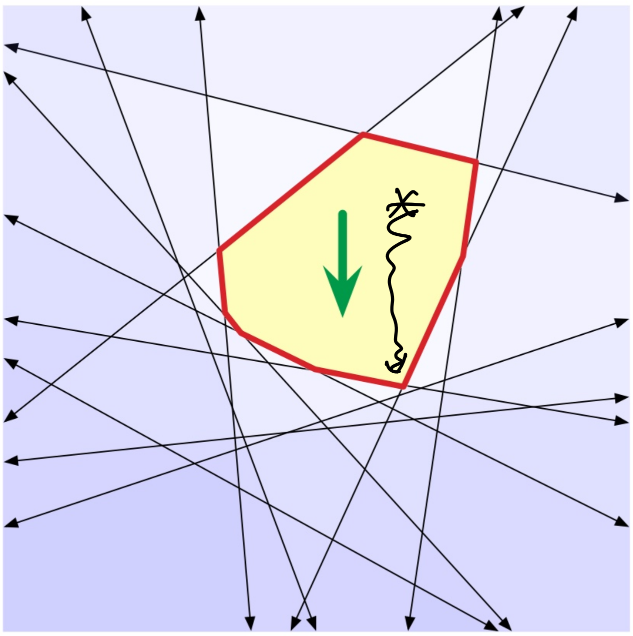


dual feasible

Basis is locally optimal if

location is optimal for LP with only the basis constraints and same objective

Primal	Dual
$\boxed{\text{optimal}}$	$\boxed{\text{optimal}}$
basis	basis
value	value
$\binom{n+d}{d}$ =	$\binom{d+n}{n}$
Feasible	loc. opt.
loc opt	Feasible
infeasible	unbounded
unbounded	infeasible

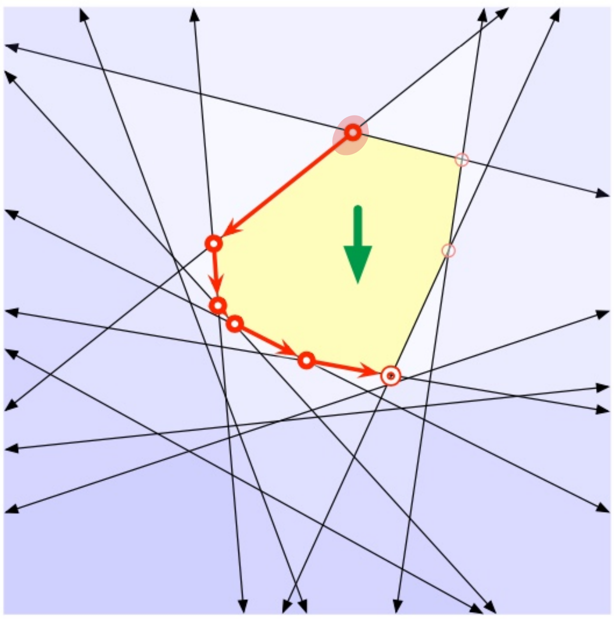


```
PrimalSimplex(H):
  if ∩H = ∅
      return Infeasible
  x ← any feasible ~~vertex~~ basis location
  while x is not locally optimal
      ⟨⟨pivot downward, maintaining feasibility⟩⟩
      if every feasible neighbor of x is higher than x
          return Unbounded
      else
          x ← any feasible neighbor of x that is lower than x
  return x
```

Magic for now

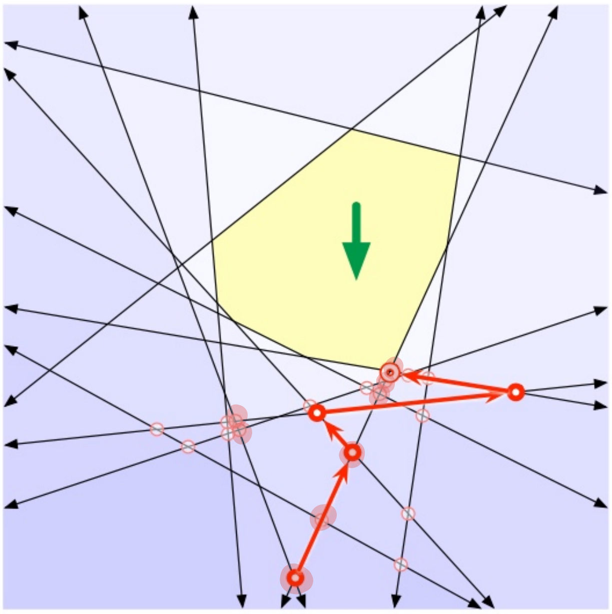
pivot

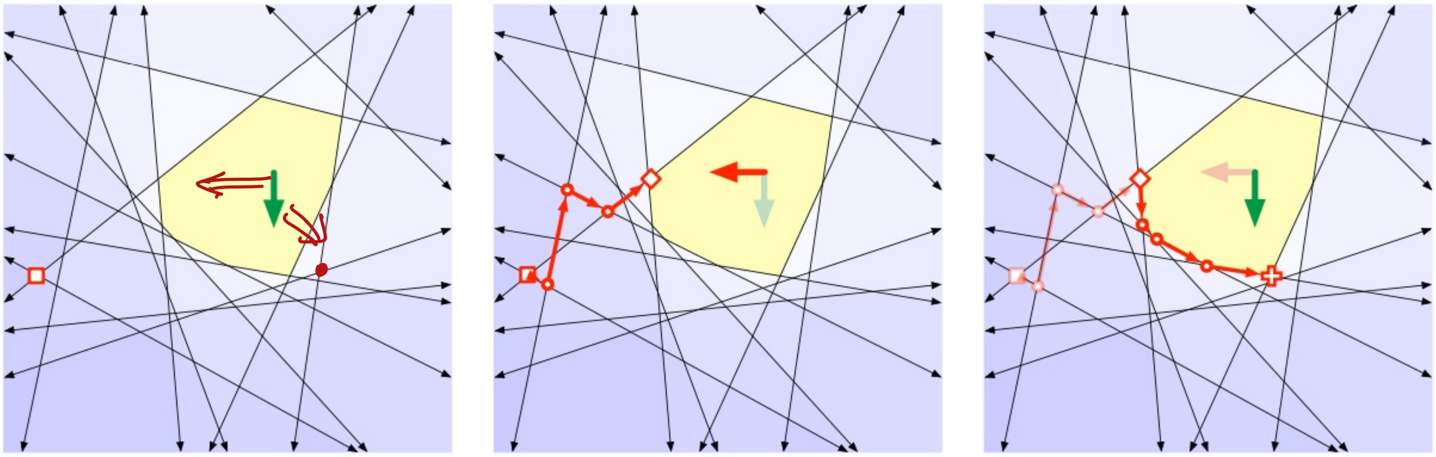
replace one constraint with another



Magic (for now)

```
DualSimplex(H):
  if there is no locally optimal vertex
      return Unbounded
  x ← any locally optimal vertex
  while x is not feasbile
      ⟨⟨pivot upward, maintaining local optimality⟩⟩
      if every locally optimal neighbor of x is lower than x
          return Infeasible
      else
          x ← any locally-optimal neighbor of x that is higher than x
  return x
```

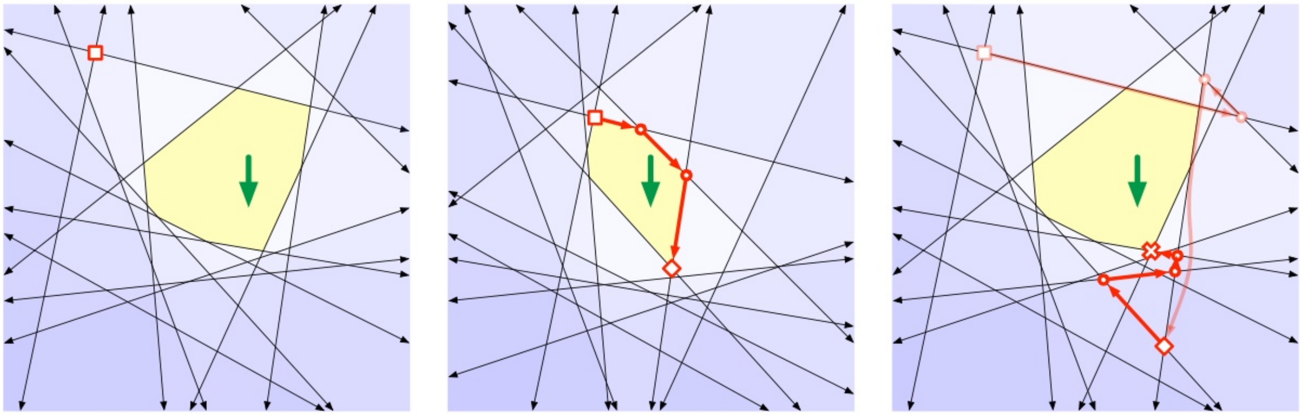




DUALPRIMALSIMPLEX(H):
    $x \leftarrow$ any vertex
    $\tilde{H} \leftarrow$ any rotation of H that makes x locally optimal   change c

    while x is not feasible
        if every locally optimal neighbor of x is lower (wrt $\tilde{H}$) than x
            return INFEASIBLE
        else
            $x \leftarrow$ any locally optimal neighbor of x that is higher (wrt $\tilde{H}$) than x

    while x is not locally optimal
        if every feasible neighbor of x is higher than x
            return UNBOUNDED
        else
            $x \leftarrow$ any feasible neighbor of x that is lower than x
    return x

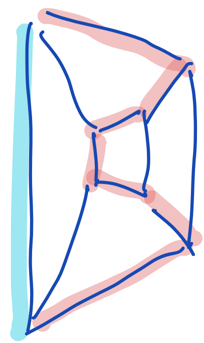


PRIMALDUALSIMPLEX(H):
    $x \leftarrow$ any vertex
    $\tilde{H} \leftarrow$ any translation of H that makes x feasible ⇐ change b
    while x is not locally optimal
        if every feasible neighbor of x is higher (wrt $\tilde{H}$) than x
            return UNBOUNDED
        else
            $x \leftarrow$ any feasible neighbor of x that is lower (wrt $\tilde{H}$) than x
    while x is not feasible
        if every locally optimal neighbor of x is lower than x
            return INFEASIBLE
        else
            $x \leftarrow$ any locally-optimal neighbor of x that is higher than x
    return x

Running time? $O(n^d)$ bleah.

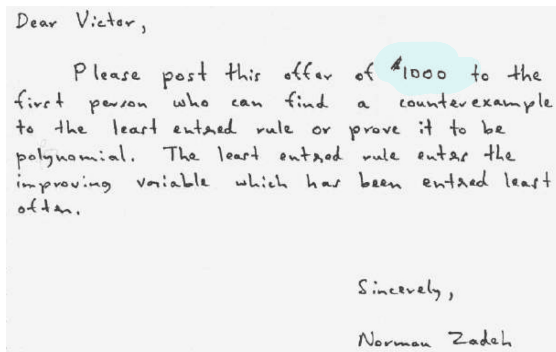
$O(n^{\lfloor d/2 \rfloor})$ primal simplex

Worst case for primal simplex is $\Theta(n^{\lfloor d/2 \rfloor})$



klee minty cubes

Every known deterministic pivot rule: exponential



Dear Victor,

Please post this offer of \$1000 to the first person who can find a counterexample to the least entered rule or prove it to be polynomial. The least entered rule enters the improving variable which has been entered least often.

Sincerely,

Norman Zadeh

randomized

subexp
superpoly

Random LP $\rightarrow$ E[time] polynomial

Any LP + noise $\rightarrow$ E[time] polynomial

$O(d^3 \log^2 n / \sigma^2)$