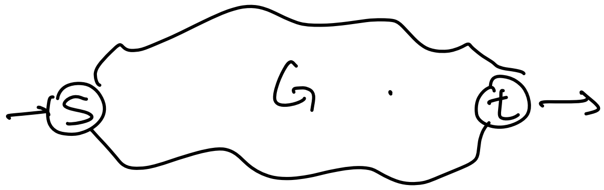
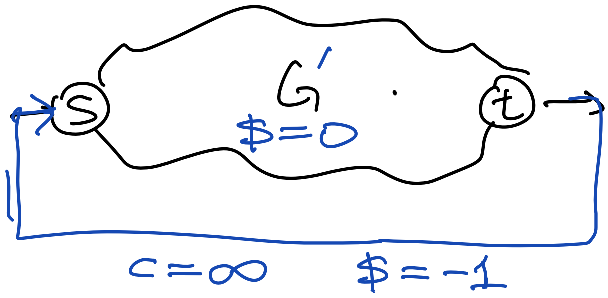
HW9 — last graded — due next Thursday

Every edge has a cost $\$(e)$

minimize cost $\$(F)$ of flow $F = \sum_e F(e) \cdot \$(e)$



max flow in G



min-cost circulation in G'

Min-cost circulation:

Input: directed graph $G = (V, E)$

Every edge e has capacity $c(e) \geq 0$
cost $\$(e)$

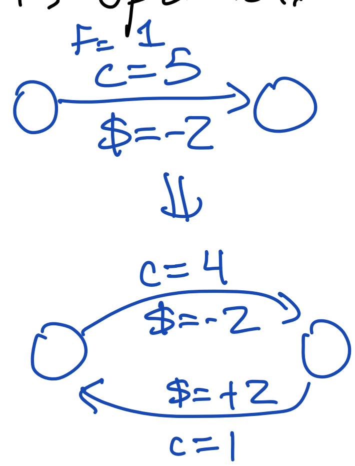
Output: $F : E \to \mathbb{R}$

$\sum_u F(u \to v) = \sum_w F(v \to w)$ for every node v

$0 \leq F(e) \leq c(e)$

$\$(F) = \sum_e F(e) \cdot \$(e)$ is minimized

If $\$(e) \geq 0$ for every edge $\Longrightarrow$ $F \equiv 0$ is optimal.



CycleCancelling$(G, c, \$)$

$F \leftarrow 0$

while G_F has a neg. cost cycle

$C \leftarrow$ any neg. cost cycle

$F \leftarrow F + C \cdot \left(\min_{e \in C} c_F(e)\right)$

return F

Bellman-Ford → $O(VE)$

If algo halts it returns min-cost circulation

Integer cap
Integer costs $\Rightarrow O(\underbrace{V \cdot E}_{\text{neg cycle}} \cdot |\$(f^*)|)$

Heuristics for good cycles:

- most neg cycle $\min \$(C)$ NP-hard
- shortest neg cycle NP-hard
- Min mean cycle $\frac{\$(C)}{\#\text{edges}(C)}$

$O(VE)$ time [Karp]

$O(V \log V)$ iterations

$\Rightarrow O(V^2 E \log V)$ [Goldberg Tarjan]

Min-cost flow in full generality:

Dir. Graph $G=(V,E)$

- capacity $c(e) \geq 0$
- lower bounds $0 \leq \ell(e) \leq c(e)$
- costs $\$(e)$
- balances $b(v)$ $\sum_v b(v) = 0$

Output: $f: E \to \mathbb{R}$

Feasible: $\ell(e) \leq f(e) \leq c(e)$ $\forall e$

balanced flow: $\sum_u f(u \to v) - \sum_w f(v \to w) = b(v)$ $\forall v$

min cost: minimize $\$(f) = \sum_e f(e) \cdot \$(e)$

Reduce to min-cost circulation

$f=0 \longrightarrow f' = \ell$ (feasible) $\xrightarrow[\text{supply} \to \text{demand paths (FF)}]{\text{Augment along}} \boxed{f''}$ (feasible + balanced) $\xrightarrow{\text{min cost circ}} f'''$

Priorities
1. Feasible
2. balanced
3. min-cost — locally optimal: no negative residual cycles

Strategy #2: Successive shortest paths

1 Feasible — $F = \ell$

2 locally optimal — $f(e) = \begin{cases} c(e) & \text{if } \$(e) < 0 \\ 0 & \text{otherwise} \end{cases}$

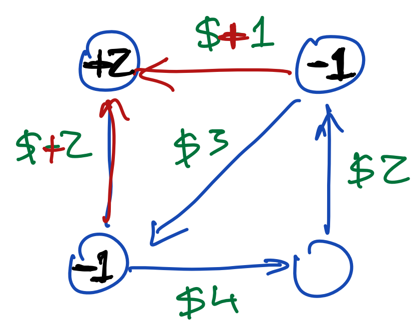
3 balanced

~~$\ell(e) = 0$~~

$\$(e) \geq 0$

$b(v)$ anything

$c(e) \geq 0$



SSP(G, b, c, $):

$f \leftarrow 0$

while f is not balanced

$s \leftarrow$ any node in G_f with $b_f(s) < 0$

$t \leftarrow$ any node in G_f with $b_f(t) > 0$

Bellman-Ford $O(VE)$ → $\sigma \leftarrow$ shortest path (wrt $\$_f$) from s to t

$f \leftarrow f + \sigma \cdot \left(\min_{e \in \sigma} c(e)\right)$

return f.

$\boxed{\text{Claim:}}$ Never neg cycles in G_f

f is always locally optimal

$\Rightarrow$ at end, f is feasible, balanced, locopt $\Rightarrow$ optimal

Proof: by induction on #iterations

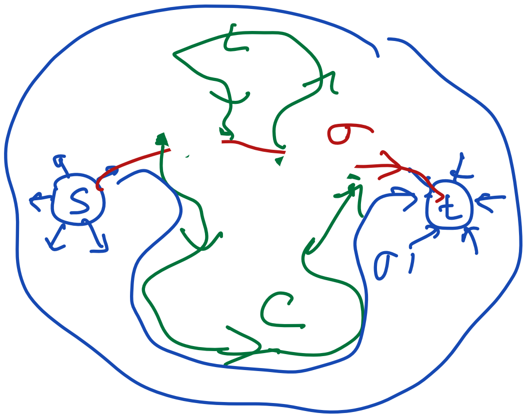
let f be flow before some iteration

f' be flow after that iteration

$f' = f + \sigma \cdot \min c(\sigma)$

IH: G_f has no neg cycles $\rightarrow$ σ is well-defined!

Spose $G_{f'}$ has neg cycle C



reverse of some edge(s) in C must be in σ

$\sigma + C$ is an (s,t)-flow with value 1

$\downarrow$ decompose

path σ' from s to t + cycles (maybe)

$\$(\sigma') \le \underline{\$(C)} + \$(\sigma) < \$(\sigma)$

Contradiction! $\square$

$O(VEB) \quad B = \sum_v |b(v)|$

$\boxed{O(E^2 \log^2 V) \text{ time}}$