Chapter 12

Approximation algorithms III

By Sariel Har-Peled, December 2, 2021[①] Version: 0.21

12.1. Clustering

Consider the problem of ***unsupervised learning***. We are given a set of examples, and we would like to partition them into classes of similar examples. For example, given a webpage X about "The reality dysfunction", one would like to find all webpages on this topic (or closely related topics). Similarly, a webpage about "All quiet on the western front" should be in the same group as webpage as "Storm of steel" (since both are about soldier experiences in World War I).

The hope is that all such webpages of interest would be in the same cluster as X, if the clustering is good.

More formally, the input is a set of examples, usually interpreted as points in high dimensions. For example, given a webpage W, we represent it as a point in high dimensions, by setting the ith coordinate to 1 if the word w_i appears somewhere in the document, where we have a prespecified list of $10,000$ words that we care about. Thus, the webpage W can be interpreted as a point of the $\{0,1\}^{10,000}$ hypercube; namely, a point in $10,000$ dimensions.

Let X be the resulting set of n points in d dimensions.

To be able to partition points into similar clusters, we need to define a notion of similarity. Such a similarity measure can be any distance function between points. For example, consider the "regular" Euclidean distance between points, where

$$\|p - q\| = \sqrt{\sum_{i=1}^{d} (p_i - q_i)^2},$$

where $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$.

As another motivating example, consider the ***facility location problem***. We are given a set X of n cities and distances between them, and we would like to build k hospitals, so that the maximum distance of a city from its closest hospital is minimized. (So that the maximum time it would take a patient to get to the its closest hospital is bounded.)

Intuitively, what we are interested in is selecting good representatives for the input point-set X. Namely, we would like to find k points in X such that they represent X "well".

Formally, consider a subset S of k points of X, and a p a point of X. The ***distance of p from the set S*** is

$$\mathbf{d}(p, S) = \min_{q \in S} \|p - q\|;$$

namely, $\mathbf{d}(p, S)$ is the minimum distance of a point of S to p. If we interpret S as a set of centers then $\mathbf{d}(p, S)$ is the distance of p to its closest center.

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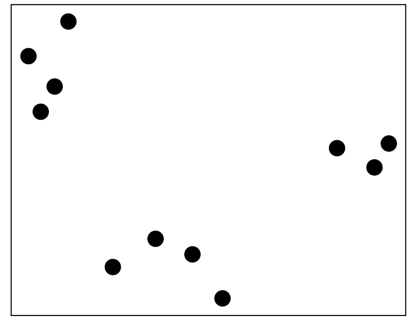
Now, the ***price of clustering*** X by the set S is

$$\nu(X, S) = \max_{p \in X} \mathbf{d}(p, S).$$



This is the maximum distance of a point of X from its closest center in S.

It is somewhat illuminating to consider the problem in the plane. We have a set P of n points in the plane, we would like to find k smallest discs centered at input points, such that they cover all the points of P. Consider the example on the right.

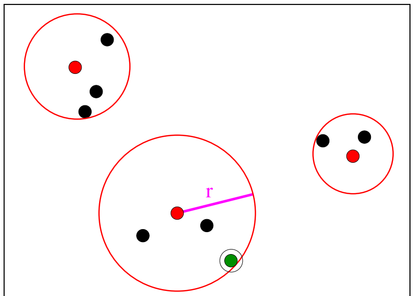


Figure 12.1: The marked point is the bottleneck point.

In this example, assume that we would like to cover it by 3 disks. One possible solution is being shown in Figure 12.1. The quality of the solution is the radius r of the largest disk. As such, the clustering problem here can be interpreted as the problem of computing an optimal cover of the input point set by k discs/balls of minimum radius. This is known as the ***k-center*** problem.

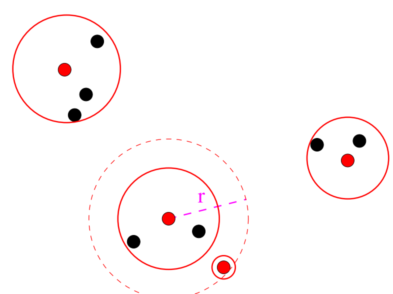
It is known that k-center clustering is NP-Hard, even to approximate within a factor of (roughly) 1.8. Interestingly, there is a simple and elegant 2-approximation algorithm. Namely, one can compute in polynomial time, k centers, such that they induce balls of radius at most twice the optimal radius.

Here is the formal definition of the k-center clustering problem.

k-center clustering

Instance: A set P a of n points, a distance function $\mathbf{d}(p, q)$, for $p, q \in P$, with triangle inequality holding for $\mathbf{d}(\cdot, \cdot)$, and a parameter k.
Question: A subset S that realizes $r_{opt}(P, k) = \min_{S \subseteq P, |S|=k} \mathsf{D}_S(P)$, where $\mathsf{D}_S(P) = \max_{x \in X} \mathbf{d}(x, S)$ and $\mathbf{d}(x, S) = \min_{s \in S} \mathbf{d}(s, x)$.

12.1.1. The approximation algorithm for k-center clustering



To come up with the idea behind the algorithm, imagine that we already have a solution with $m = 3$ centers. We would like to pick the next $m+1$ center. Inspecting the examples above, one realizes that the solution is being determined by a bottleneck point; see Figure 12.1. That is, there is a single point which determine the quality of the clustering, which is the point furthest away from the set of centers. As such, the natural step is to find a new center that would better serve this bottleneck point. And, what can be a better service for this point, than make it the next center? (The resulting clustering using the new center for the example is depicted on the right.)

Namely, we always pick the bottleneck point, which is furthest away for the current set of centers, as the next center to be added to the solution.

The resulting approximation algorithm is depicted on the right. Observe, that the quantity r_{i+1} denotes the (minimum) radius of the i balls centered at $u_1, \ldots, u_i$ such that they cover P (where all these balls have the same radius). (Namely, there is a point $p \in P$ such that $\mathbf{d}(p, \{u_1, \ldots, u_i\}) = r_{i+1}$.

It would be convenient, for the sake of analysis, to imagine that we run AprxKCenter one additional iteration, so that the quantity r_{k+1} is well defined.

Observe, that the running time of the algorithm AprxKCenter is $O(nk)$ as can be easily verified.

```
AprxKCenter(P, k)
  P = {p_1, ..., p_n}
  S = {p_1}, u_1 ← p_1
  while |S| < k do
    i ← |S|
    for j = 1 ... n do
      d_j ← min(d_j, d(p_j, u_i))
    r_{i+1} ← max(d_1, ..., d_n)
    u_{i+1} ←point of P realizing r_i
    S ← S ∪ {u_{i+1}}

  return S
```

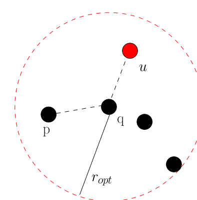
Lemma 12.1.1. *We have that $r_2 \geq \ldots \geq r_k \geq r_{k+1}$.*

Proof: At each iteration the algorithm adds one new center, and as such the distance of a point to the closest center can not increase. In particular, the distance of the furthest point to the centers does not increase. ∎

Observation 12.1.2. *The radius of the clustering generated by AprxKCenter is r_{k+1}.*

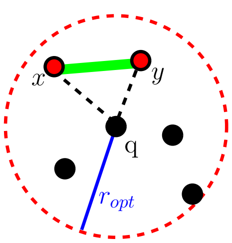
Lemma 12.1.3. *We have that $r_{k+1} \leq 2r_{opt}(P, k)$, where $r_{opt}(P, k)$ is the radius of the optimal solution using k balls.*

Proof: Consider the k balls forming the optimal solution: $D_1, \ldots, D_k$ and consider the k center points contained in the solution S computed by AprxKCenter.



If every disk D_i contain at least one point of S, then we are done, since every point of P is in distance at most $2r_{opt}(P, k)$ from one of the points of S. Indeed, if the ball D_i, centered at q, contains the point $u \in S$, then for any point $p \in P \cap D_i$, we have that

$$\mathbf{d}(p, u) \leq \mathbf{d}(p, q) + \mathbf{d}(q, u) \leq 2r_{opt}.$$



Otherwise, there must be two points x and y of S contained in the same ball D_i of the optimal solution. Let D_i be centered at a point q.

We claim distance between x and y is at least r_{k+1}. Indeed, imagine that x was added at the αth iteration (that is, $u_\alpha = x$), and y was added in a later βth iteration (that is, $u_\beta = y$), where $\alpha < \beta$. Then,

$$r_\beta = \mathbf{d}\big(y, \{u_1, \ldots, u_{\beta-1}\}\big) \leq \mathbf{d}(x, y),$$

since $x = u_\alpha$ and $y = u_\beta$. But $r_\beta \geq r_{k+1}$, by Lemma 12.1.1. Applying the triangle inequality again, we have that $r_{k+1} \leq r_\beta \leq \mathbf{d}(x, y) \leq \mathbf{d}(x, q) + \mathbf{d}(q, y) \leq 2r_{opt}$, implying the claim. ∎

Theorem 12.1.4. *One can approximate the k-center clustering up to a factor of two, in time $O(nk)$.*

Proof: The approximation algorithm is AprxKCenter. The approximation quality guarantee follows from Lemma 12.1.3, since the furthest point of P from the k-centers computed is r_{k+1}, which is guaranteed to be at most $2r_{opt}$. ∎

12.2. Subset Sum

Subset Sum

> **Instance**: $X = \{x_1, \ldots, x_n\}$ – n integer positive numbers, t - target number
> **Question:** Is there a subset of X such the sum of its elements is t?

Subset Sum is (of course) NPC, as we already proved. It can be solved in polynomial time if the numbers of X are small. In particular, if $x_i \leq M$, for $i = 1, \ldots, n$, then $t \leq Mn$ (otherwise, there is no solution). Its reasonably easy to solve in this case, as the algorithm on the right shows. The running time of the resulting algorithm is $O(Mn^2)$.

```
SolveSubsetSum (X, t, M)
   b[0 ... Mn] - boolean array init to FALSE.
      // b[x] is TRUE if x can be realized by
          a subset of X.
   b[0] ← TRUE.
   for i = 1, ..., n do
      for j = Mn down to x_i do
         b[j] ← B[j − x_i] ∨ B[j]

   return B[t]
```

Note, that M might be prohibitly large, and as such, this algorithm is not polynomial in n. In particular, if $M = 2^n$ then this algorithm is prohibitly slow. Since the relevant decision problem is NPC, it is unlikely that an efficient algorithm exist for this problem. But still, we would like to be able to solve it quickly and efficiently. So, if we want an efficient solution, we would have to change the problem slightly. As a first step, lets turn it into an optimization problem.

Subset Sum Optimization

> **Instance**: (X, t): A set X of n positive integers, and a target number t.
> **Question:** The largest number γ_{opt} one can represent as a subset sum of X which is smaller or equal to t.

Intuitively, we would like to find a subset of X such that it sum is smaller than t but very close to t. Next, we turn problem into an approximation problem.

Subset Sum Approx

> **Instance**: (X, t, ε): A set X of n positive integers, a target number t, and parameter $\varepsilon > 0$.
> **Question:** A number z that one can represent as a subset sum of X, such that $(1 - \varepsilon)\gamma_{\mathrm{opt}} \leq z \leq \gamma_{\mathrm{opt}} \leq t$.

The challenge is to solve this approximation problem efficiently. To demonstrate that there is hope that can be done, consider the following simple approximation algorithm, that achieves a constant factor approximation.

Lemma 12.2.1. *Let (X, t) be an instance of Subset Sum. Let γ_{opt} be optimal solution to given instance. Then one can compute a subset sum that adds up to at least $\gamma_{\mathrm{opt}}/2$ in $O(n \log n)$ time.*

Proof: Add the numbers from largest to smallest, whenever adding a number will make the sum exceed t, we throw it away. We claim that the generated sum s has the property that $\gamma_{\mathrm{opt}}/2 \leq s \leq t$. Clearly, if the total sum of the numbers is smaller than t, then no number is being rejected and $s = \gamma_{\mathrm{opt}}$.

Otherwise, let u be the first number being rejected, and let s' be the partial subset sum, just before u is being rejected. Clearly, $s' > u > 0$, $s' < t$, and $s' + u > t$. This implies $t < s' + u < s' + s' = 2s'$, which implies that $s' \geq t/2$. Namely, the subset sum output is larger than $t/2$. ■

12.2.1. On the complexity of ε-approximation algorithms

Definition 12.2.2 (PTAS). For a maximization problem PROB, an algorithm $\mathcal{A}(I, \varepsilon)$ (i.e., A receives as input an instance of PROB, and an approximation parameter $\varepsilon > 0$) is a ***polynomial time approximation scheme*** (***PTAS***) if for any instance I we have

$$(1 - \varepsilon) \left|\mathsf{opt}(I)\right| \leq \left|\mathcal{A}(I, \varepsilon)\right| \leq \left|\mathsf{opt}(I)\right|,$$

where $|\mathsf{opt}(I)|$ denote the price of the optimal solution for I, and $|\mathcal{A}(I, \varepsilon)|$ denotes the price of the solution outputted by $\mathcal{A}$. Furthermore, the running time of the algorithm $\mathcal{A}$ is polynomial in n (the input size), when ε is fixed.

For a minimization problem, the condition is that $|\mathsf{opt}(I)| \leq |\mathcal{A}(I, \varepsilon)| \leq (1 + \varepsilon)|\mathsf{opt}(I)|$.

Example 12.2.3. An approximation algorithm with running time $O(n^{1/\varepsilon})$ is a PTAS, while an algorithm with running time $O(1/\varepsilon^n)$ is not.

Definition 12.2.4 (FPTAS.). An approximation algorithm is ***fully polynomial time approximation scheme*** (***FPTAS***) if it is a PTAS, and its running time is polynomial both in n and $1/\varepsilon$.

Example 12.2.5. A PTAS with running time $O(n^{1/\varepsilon})$ is not a FPTAS, while a PTAS with running time $O(n^2/\varepsilon^3)$ is a FPTAS.

12.2.2. Approximating subset-sum

Let $S = \{a_1, \ldots, a_n\}$ be a set of numbers. For a number x, let $x + S$ denote the translation of S by x; namely, $x + S = \{a_1 + x, a_2 + x, \ldots a_n + x\}$. Our first step in deriving an approximation algorithm for Subset Sum is to come up with a slightly different algorithm for solving the problem exactly. The algorithm is depicted on the right.

```
ExactSubsetSum(S, t)
    n ← |S|
    P_0 ← {0}
    for i = 1 ... n do
        P_i ← P_{i-1} ∪ (P_{i-1} + x_i)
        Remove from P_i all elements > t

    return largest element in P_n
```

Note, that while ExactSubsetSum performs only n iterations, the lists P_i that it constructs might have exponential size.

```
Trim(L', δ)
    // L': inc. sorted list of #s
    L = ⟨y_1 ... y_m⟩
        // y_i ≤ y_{i+1}, for i = 1, ..., n − 1.
    curr ← y_1
    L_out ← {y_1}
    for i = 2 ... m do
        if y_i > curr · (1 + δ)
            Append y_i to L_out
            curr ← y_i
    return L_out
```

Thus, if we would like to turn ExactSubsetSum into a faster algorithm, we need to somehow to make the lists L_l smaller. This would be done by removing numbers which are very close together.

Definition 12.2.6. For two positive real numbers $z \leq y$, the number y is a δ-approximation to z if $\dfrac{y}{1+\delta} \leq z \leq y$.

The procedure Trim that trims a list L' so that it removes close numbers is depicted on the left.

Observation 12.2.7. *If $x \in L'$ then there exists a number $y \in L_{out}$ such that $y \leq x \leq y(1 + \delta)$, where $L_{out} \leftarrow$ Trim(L', δ).*

We can now modify ExactSubsetSum to use Trim to keep the candidate list shorter. The resulting algorithm ApproxSubsetSum is depicted on the right. Note, that computing E_i requires merging two sorted lists, which can be done in linear time in the size of the lists (i.e., we can keep all the lists sorted, without sorting the lists repeatedly).

Let E_i be the list generated by the algorithm in the ith iteration, and P_i be the list of numbers without any trimming (i.e., the set generated by ExactSubsetSum algorithm) in the ith iteration.

```
ApproxSubsetSum(S, t)
   //Assume S = {x_1, ..., x_n}, where
   //      x_1 ≤ x_2 ≤ ... ≤ x_n
   n ← |S|, L_0 ← {0}, δ = ε/2n
   for i = 1 ... n do
      E_i ← L_{i-1} ∪ (L_{i-1} + x_i)
      L_i ← Trim(E_i, δ)
      Remove from L_i all elements > t.

   return largest element in L_n
```

Claim 12.2.8. *For any $x \in P_i$ there exists $y \in L_i$ such that $y \le x \le (1+\delta)^i y$.*

Proof: If $x \in P_1$ the claim follows by Observation 12.2.7 above. Otherwise, if $x \in P_{i-1}$, then, by induction, there is $y' \in L_{i-1}$ such that $y' \le x \le (1+\delta)^{i-1} y'$. Observation 12.2.7 implies that there exists $y \in L_i$ such that $y \le y' \le (1+\delta)y$, As such,

$$y \le y' \le x \le (1+\delta)^{i-1} y' \le (1+\delta)^i y$$

as required.

The other possibility is that $x \in P_i \setminus P_{i-1}$. But then $x = \alpha + x_i$, for some $\alpha \in P_{i-1}$. By induction, there exists $\alpha' \in L_{i-1}$ such that

$$\alpha' \le \alpha \le (1+\delta)^{i-1} \alpha'.$$

Thus, $\alpha' + x_i \in E_i$ and by Observation 12.2.7, there is a $x' \in L_i$ such that

$$x' \le \alpha' + x_i \le (1+\delta) x'.$$

Thus,

$$x' \le \alpha' + x_i \le \alpha + x_i = x \le (1+\delta)^{i-1}\alpha' + x_i \le (1+\delta)^{i-1}(\alpha' + x_i) \le (1+\delta)^i x'.$$

Namely, for any $x \in P_i \setminus P_{i-1}$, there exists $x' \in L_i$, such that $x' \le x \le (1+\delta)^i x'$. ■

12.2.2.1. Bounding the running time of ApproxSubsetSum

We need the following two easy technical lemmas. We include their proofs here only for the sake of completeness.

Lemma 12.2.9. *For $x \in [0,1]$, it holds $\exp(x/2) \le (1+x)$.*

Proof: Let $f(x) = \exp(x/2)$ and $g(x) = 1 + x$. We have $f'(x) = \exp(x/2)/2$ and $g'(x) = 1$. As such,

$$f'(x) = \frac{\exp(x/2)}{2} \le \frac{\exp(1/2)}{2} \le 1 = g'(x), \qquad \text{for } x \in [0,1].$$

Now, $f(0) = g(0) = 1$, which immediately implies the claim. ■

Lemma 12.2.10. *For $0 < \delta < 1$, and $x \ge 1$, we have $\log_{1+\delta} x \le \dfrac{2\ln x}{\delta} = O\left(\dfrac{\ln x}{\delta}\right)$.*

Proof: We have, by Lemma 12.2.9, that $\log_{1+\delta} x = \dfrac{\ln x}{\ln(1+\delta)} \leq \dfrac{\ln x}{\ln \exp(\delta/2)} = \dfrac{2 \ln x}{\delta}$. ■

Observation 12.2.11. *In a list generated by Trim, for any number x, there are no two numbers in the trimmed list between x and $(1+\delta)x$.*

Lemma 12.2.12. *We have $|L_i| = O\left(\dfrac{n^2}{\varepsilon} \log n\right)$, for $i = 1, \ldots, n$.*

Proof: The set $L_{i-1} + x_i$ is a set of numbers between x_i and $i x_i$, because x_i is larger than $x_1 \ldots x_{i-1}$ and L_{i-1} contains subset sums of at most $i-1$ numbers, each one of them smaller than x_i. As such, the number of different values in this range, stored in the list L_i, after trimming is at most

$$\log_{1+\delta} \frac{i x_i}{x_i} = O\left(\frac{\ln i}{\delta}\right) = O\left(\frac{\ln n}{\delta}\right),$$

by Lemma 12.2.10. Thus, as $\delta = \varepsilon/2n$, we have

$$|L_i| \leq |L_{i-1}| + O\left(\frac{\ln n}{\delta}\right) \leq |L_{i-1}| + O\left(\frac{n \ln n}{\varepsilon}\right) = O\left(\frac{n^2 \log n}{\varepsilon}\right).$$ ■

Lemma 12.2.13. *The running time of ApproxSubsetSum is $O\left(\frac{n^3}{\varepsilon} \log^2 n\right)$.*

Proof: Clearly, the running time of ApproxSubsetSum is dominated by the total length of the lists $L_1, \ldots, L_n$ it creates. Lemma 12.2.12 implies that $\sum_i |L_i| = O\left(\dfrac{n^3}{\varepsilon} \log n\right)$. The running time of Trim is proportional to the size of the lists, implying the claimed running time. ■

12.2.2.2. The result

Theorem 12.2.14. *ApproxSubsetSum returns a number $u \leq t$, such that*

$$\frac{\gamma_{\mathrm{opt}}}{1+\varepsilon} \leq u \leq \gamma_{\mathrm{opt}} \leq t,$$

where γ_{opt} is the optimal solution (i.e., largest realizable subset sum smaller than t).

The running time of ApproxSubsetSum is $O\big((n^3/\varepsilon) \log n\big)$.

Proof: The running time bound is by Lemma 12.2.13.

As for the other claim, consider the optimal solution $\mathsf{opt} \in P_n$. By Claim 12.2.8, there exists $z \in L_n$ such that $z \leq \mathsf{opt} \leq (1+\delta)^n z$. However,

$$(1+\delta)^n = (1+\varepsilon/2n)^n \leq \exp\left(\frac{\varepsilon}{2}\right) \leq 1+\varepsilon,$$

since $1+x \leq e^x$ for $x \geq 0$. Thus, $\mathsf{opt}/(1+\varepsilon) \leq z \leq \mathsf{opt} \leq t$, implying that the output of ApproxSubsetSum is within the required range. ■

12.3. Approximate Bin Packing

Consider the following problem.

Min Bin Packing

> **Instance**: $s_1 \ldots s_n$ – n numbers in $[0, 1]$
> **Question:** Q: What is the minimum number of unit bins do you need to use to store all the numbers in S?

Bin Packing is NP-Complete because you can reduce Partition to it. Its natural to ask how one can approximate the optimal solution to Bin Packing.

One such algorithm is ***next fit***. Here, we go over the numbers one by one, and put a number in the current bin if that bin can contain it. Otherwise, we create a new bin and put the number in this bin. Clearly, we need at least

$$\lceil S \rceil \text{ bins where } S = \sum_{i=1}^{n} s_i.$$

Every two consecutive bins contain numbers that add up to more than 1, since otherwise we would have not created the second bin. As such, the number of bins used is $\leq 2\lceil S \rceil$. As such, the next fit algorithm for bin packing achieves a $\leq 2\lceil S \rceil / \lceil S \rceil = 2$ approximation.

A better strategy, is to sort the numbers from largest to smallest and insert them in this order, where in each stage, we scan all current bins, and see if can insert the current number into one of those bins. If we can not, we create a new bin for this number. This is known as ***first fit decreasing***. We state the approximation ratio for this algorithm without proof.

Theorem 12.3.1 ([DLHT13]). *Decreasing first fit is a $11/9$-approximation to Min Bin Packing. More precisely, for any instance I of the problem, one has*

$$\mathrm{FFD}(I) \leq \frac{11}{9}\mathrm{opt}(I) + \frac{2}{3},$$

and this is tight in the worst case. Here $\mathrm{FFD}(I)$ and $\mathrm{opt}(I)$ are the number of bins used by the first-fit decreasing algorithm and optimal solution, respectively.

Remark 12.3.2. Note that if $\mathrm{opt}(I) = 2$, then the above bound is $\mathrm{FFD}(I) \leq \frac{11}{9}2 + \frac{2}{3} = \frac{28}{9} = 3\frac{1}{9}$, Which means that in this case this approach could yield a solution with three bins, which is not exciting.

The above paper is almost 50 pages long, and is not easy. The coefficient 11/9 was proved by David S. Johnson in his PhD thesis in 1973 (who also authored [GJ90]), but the exact value of the additive constant was only settled by [DLHT13].

Remark 12.3.3. Note, that the above algorithm is not a multiplicative approximation (note the $+2/3$ term). In particular, getting a 3/2-approximation is hard because of the reduction from Partition – there the decision boils down to whether the instance generated from partition requires two bins or three bins. As such, any multiplicative approximation better than 3/2 is impossible unless $\mathrm{P} = \mathrm{NP}$.

12.4. Bibliographical notes

One can do 2-approximation for the k-center clustering in low dimensional Euclidean space can be done in $\Theta(n \log k)$ time [FG88]. In fact, it can be solved in linear time [Har04].

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