Fair Division

↳ Dividing "items" among agents in a "fair" and "efficient" manner

→ Early mention → Bible
"cut & choose" → land division Abraham & Lot

→ Application →
1) Division of labor / chore
2) Division of property
   ↳ divorce
   ↳ settling inheritance
   ↳ dissolving partner
3) Distribut of vaccines / health care
4) Air traffic - man

SP21 DDM.

Setup. (Discrete fair division)

Given: → 1) set N of n agents
2) set M of m indivisible goods.
3) Each agent has a valuation fn.
$v_i : 2^M \to \mathbb{R}_{\geq 0}$

weak-monotonicity $v_i(S \cup \{g\}) \geq v_i(S)$ $\forall i \in N$

Additive valuations : $v_i(S) = \sum_{g \in S} v_i(\{g\})$

Find :- Partition $\langle X_1, X_2, X_i \ldots X_n \rangle$ of M

X_i allocated to agent i

"fair" & "efficient".

Fairness.

"Envy-Freeness" :-

agent i envies agent j. $\quad v_i(X_i) < v_i(X_j)$

$\forall i, j \qquad v_i(X_i) \geq v_i(X_j)$

"Pareto optimal" :- An allocation X is pareto-optimal if there exists no allocation Y, where $v_i(Y_i) \geq v_i(X_i) \;\forall i \in N$, with strict inequality for at least one agent

	g_1	g_2	g_3	g_4
a_1	1	1	0	0
a_2	0	0	1	1

$X_1 \leftarrow \{g_1, g_3\}$

$X_2 \leftarrow \{g_2, g_4\}$ EF

$v_1(\{g_1, g_3\}) = 1 = v(\{g_2, g_4\})$

Q-2) Does EF exist always?

Relaxations of EF

Envy-freeness upto one good (EF1): x is EF1 iff

only if $\forall\, i,j$ $\quad v_i(x_i) \geq v_i(x_j \setminus \{g\})$ for some $g \in x_j$.

	laptop	iPad	phone
a_1	10	6	9
a_2	10	6	3

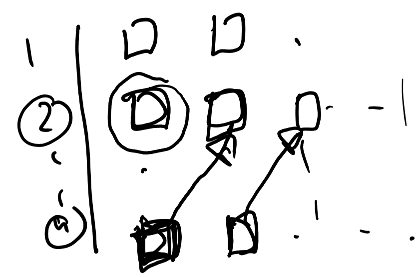
EF1

$$v_1(\{\text{laptop}\}) \geq v_1(\{\text{iPad}\})$$
$$= v_1(x_2 \setminus \{\text{phone}\})$$
$$v_2(\{\text{iPad, phone}\}) \geq v_2(x_1 \setminus \{\text{laptop}\})$$

Does EF1 allocations always exist?

Round-Robin algorithm

1) order agents $[1, \dots, n]$
2) while $\exists$ good left
   every agent picks his most desired good in every round in the order



Claim: RR returns an EF1 allocation.

if $i < j$, $\quad v_i(x_i) \geq v_i(x_j)$

h_i: good allocated to agent i in round 1

$\forall j > i, \quad v_j(x_j) \geq v_j(x_i \setminus \{h_i\})$

1) EF1 allocations exist & can be found in polynomial time

Does EF1 + PO allocations exist?

$$\phi(x_1, x_2, \ldots x_n) = \underbrace{\left(\prod_{i \in [n]} v_i(x_i)\right)^{1/n}}_{\text{Nash-welfare of an allocation}}$$

Thm: Any allocation that maximizes Nash-welfare is EF1 & PO.

$\exists\, i$ & $j \qquad v_i(x_i) < v_i(x_j \setminus \{g\})$ for all $g \in x_j$.

X'

$$x'_k = \begin{cases} x_k & \forall k \notin \{i, j\} \\ x_i \cup \{g\} & \\ x_j \setminus \{g\} & \text{where } g \in x_j \text{ with maximum value of } \frac{v_i g}{\ldots} \end{cases}$$

v_{jg}

OPEN-PROBLEM-

Does there exist a polynomial time alg to find EF1 + PO.?

$(1-\varepsilon)$ EF1 + PO - in polynomial time

$\forall i, j \quad v_i(x_i) \geq (1-\varepsilon) \cdot v_i(x_j \setminus \{g\})$

~~for all~~ for some $g \in x_j$

"Proportionality" Ex: $\langle x_1, x_2, \ldots, x_n \rangle$ is proportional

$$v_i(x_i) \geq \frac{1}{n} \cdot v_i(\underline{M})$$

Recall cut & choose

Max-min-share :- Given N, M, the max-min-share of an agent $MMS_i(N,M)$

$$MMS_i(N,M) = \max_{\substack{X = \langle x_1, x_2, \ldots x_n \rangle \\ \in \mathcal{X}}} \min_{j \in [n]} v_i(X_j)$$

An allocation $x = \langle x_1, x_2, \ldots, x_n \rangle$ is fair iff $\quad v_i(x_i) \geq MMS_i(N,M)$

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claim :- $MMS_i(N,M) \leq v_i(M)$

Claim: $MMS_i(N, M) \leq \frac{v_i(M)}{n}$.

Cut & choose works for 2 agents

Agent 1 does his MMS optimal partition

X_1 & X_2 $\quad v_1(X_1) \geq MMS_1$

$v_1(X_2) \geq MMS_1$

Agent 2 chooses X_2

$\Rightarrow v_2(X_2) \geq v_2(X_1)$

$\Rightarrow v_2(X_2) \geq \frac{1}{2} \cdot v_2(X_1 \cup X_2)$

$= \frac{1}{2} \cdot v_2(M)$

$\geq MMS_2$

Thm: MMS do not exist even with three agents.

Approximations? $\langle X_1, X_2, \ldots, X_n \rangle$

where $v_i(X_i) \geq \alpha \cdot MMS_i(N, M)$ for some scalar α.

Thm $3/4$ - MMS allocations exist.

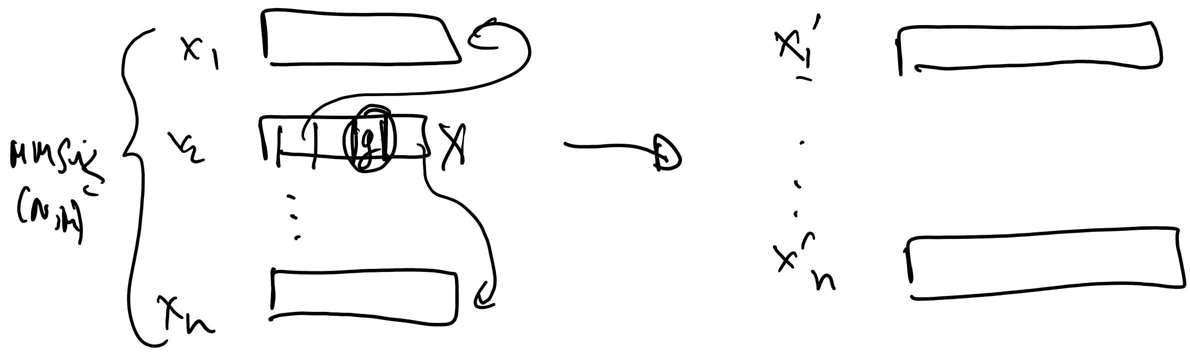
OPEN PROBLEM $> 3/4$ - - — — ?

Thm: $\frac{1}{2}$-MMS allocations exist.

Reduction:

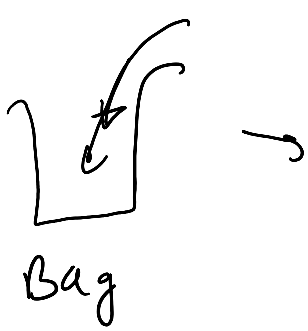
Claim 1: $MMS_i(N, M) \le MMS_i(N\setminus\{i'\}, M\setminus\{g\})$

Look at MMS optimal partition of agent i



$\le MMS_i(N\setminus\{i\}, M\setminus\{g\})$

Claim 2: $\forall i \in N,\ \forall g \in M,\ v_i(g) < \delta$
$\Rightarrow \exists\ (1-\delta)$ MMS fair partition.



As soon as bag is valued at $1-\delta$ by any agent

(1) (2) — — — (n)

$x_1 \leftarrow B_1$, $x_2 \leftarrow B_2$,

$v_i(M) = n$ # agent left unallocated $= n$.

<u>Claim</u>: $\Rightarrow$ no agent values B_1 more than 1

least good put in the bag $= g$.

$$v_i(B \setminus g) \leq 1-\delta.$$
$$+ \quad v_i(g) \leq \delta$$
$$v_i(B) \leq 1.$$

$v_i(M) \geq n-1$ # agents $= n-1$

$\Rightarrow$ All agents are allocated bundle w $1-\delta$

$\Rightarrow (1-\delta) \cdot \underline{MMS_i(N, M)}$

$\hookrightarrow \leq \frac{v_i(M)}{n} = \frac{n}{n} = 1$

<u>Alg</u> $\frac{1}{2}$-MMS (N, M)

1) while $\exists$ $\underline{i \in N, \exists g \in M}$

$$v_i(g) \geq \frac{1}{2} \cdot MMS_i(N, M)$$

$$\left.\begin{array}{l} x_i \leftarrow g. \\ \frac{1}{2}\text{-MMS}(N \setminus \{i\}, M \setminus \{g\}) \end{array}\right\} \rightarrow \frac{1}{2}\text{MMS}$$

Bag-filling $(N, M) \rightarrow$ $\frac{1}{2}$-MMS

Thm: $\Rightarrow$ $\exists$ allocations that are $\frac{1}{2}$ MMS + PO.