L 25: Approximation algorithms using LP

Vertex cover

$G = (V, E)$ undirected graph

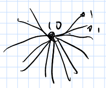
Compute smallest $X \subseteq V$ s.t.
$\forall e \in E \quad X \cap e \neq \emptyset$

Weighted version

$\forall v \in V \quad c_v > 0$ cost of the vertex

compute $X \subseteq V$ s.t. $\min \sum_{v \in X} c_v$

s.t. X is a VC in G



IP for VC

$\forall v \in V \quad x_v \in \{0,1\}$

$\min \sum_{v \in V} c_v x_v$

$\forall uv \in E \quad x_u + x_v \geq 1$

LP

$\min \sum_{v \in V} c_v x_v$

$\forall uv \in E \quad x_u + x_v \geq 1$

$\forall v \in V \quad x_v \geq 0$

opt LP

Integral solution $\geq$ Fractional solution

$\forall uv \in E \quad x_u + x_v \geq 1$

must be $\widehat{x_u} \geq 1/2$ or $\widehat{x_v} \geq 1/2$

$C = \{ v \in V \mid \widehat{x_v} \geq 1/2 \}$

claim (i) C is a VC of G.
(ii) $cost(C) \leq 2\, opt_{LP}$

Proof

$cost(C) = \sum_{v \in C} c_v \leq \sum_{v \in C} 2\widehat{x_v} c_v \leq 2 \sum_{v \in V} x_v c_v \leq 2\, opt_{LP}$

$\leq 2\, opt_{VC}$. $\square$

Theorem
Weighted VC can be 2-approximated by solving LP once, and doing rounding as described.

Set Cover

$(U, F) \quad F \subseteq 2^U$

Q: Compute min size $H \subseteq F$ s.t. $\bigcup_{f \in H} f = U$.

IP

$U = \{e_1, e_2, \dots, e_n\} \quad F = \{f_1, f_2, \dots, f_m\}$
$f_i \subseteq U$

$\min \sum_{i=1}^{m} x_i$

$x_i \in \{0,1\} \quad \forall i = 1 \dots m$

$x_i = 1 \iff f_i$ is in the cover

$\forall j = 1 \dots n$
$e_j \quad \sum_{i : e_j \in f_i} x_i \geq 1$

$\min \sum_i x_i$

$j = 1, \dots, n \quad \sum_{i : e_j \in f_i} x_i \geq 1$

$\forall i \quad x_i \geq 0$

$x_i \in [0,1]$

$v_1 \quad \sum_{i : e_1 \in f_i} \widehat{x_i} \geq 1$

$C_1 = \{ f_i \in F \mid e_1 \in f_i \}$

$R \subseteq F$ generated by picking f_i into the set with probability $\widehat{x_i}$.

$E[|C \cap R|] = E\left[\sum_{f_i \in C_1} [f_i \in R]\right]$

$= \sum_{f_i \in C_1} [f_i \in R] = \sum_{i \in C_1} \widehat{x_i} \geq 1$

Claim $P[v_i$ is not covered by $\cup R] \leq \frac{1}{2}$.
$\leq$

Proof

C_1 all sets covering v_1

$P[v_1$ is not covered by $R]$

$= P[$none of the sets in C_1 were picked in $R]$ $\qquad 1 - x \leq e^{-x}$

$= \prod_{f_i \in C_1} P[f_i \text{ was not picked}] = \prod_{f_i \in C_1} (1 - \widehat{x_i}) \leq \prod_{f_i \in C_1} \exp(-\widehat{x_i}) = \exp\left(-\sum_{f_i \in C_1} \widehat{x_i}\right) \leq \exp(-1) = \frac{1}{e} \leq \frac{1}{2}$

$1 - \widehat{x_i}$

$\sum_{f_i \in C_1} \widehat{x_i} \geq 1$

$\blacksquare$

Let $R_1, R_2, \dots R_u$ be the covers computed in $u = O(\log n)$ rounds.

Claim $\bigcup$ is over of U with prob. $1 - 1$

Proof

Claim

$\bigcup_{i=1}^{u} R_i$ is a cover of the ground set w.h.p.

Proof

$$P[X_i \text{ is not covered}] = \prod_{i=1}^{u} P[R_i \text{ does not cover } v_i]$$

$$\leq \frac{1}{2^u} \leq \frac{1}{n^{O(1)}}$$

$$P\left[\begin{matrix}\text{bad}\\ \text{event}\end{matrix}\right] = \bigcup_i P[X_i \text{ not covered}]$$

$$\leq n \cdot \frac{1}{n^{O(1)}} \leq \frac{1}{n^{O(1)}}. \quad \blacksquare$$

Claim

$$E[|R_i|] = \sum \hat{x}_i = opt_{LP}$$

$$\Rightarrow E\left[\left|\bigcup_{i=1}^{u} R_i\right|\right] \leq E\left[\sum_{i=1}^{u} |R_i|\right] = \sum_{i=1}^{u} E[|R_i|]$$

$$= opt_{LP} \cdot u$$

$$= O(opt_{LP} \cdot \log n). \quad \blacksquare$$

Congestion

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