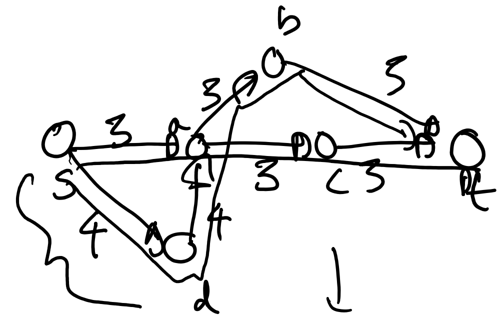
Network flows

Graph with edge-capacities

"transfer"



Flow-network :→ Graph $G = (V, E, c)$ with $c : V \times V \to \mathbb{R}_{\geq 0}$

$$c(u,v) = 0 \quad \forall (u,v) \notin E$$

Flow :→ A function $f : V \times V \to \mathbb{R}$

1) **Capacity-constraint** → $f(u,v) \leq c(u,v) \quad \forall u,v$
2) **Anti-symmetry** :→ $f(u,v) = -f(v,u) \quad \forall u,v$
3) **Flow-conservation** :→ total inflow = total out flow

$$f(V,V) = 0 \quad \Leftarrow \quad \sum_{v \in V} f(u,v) = 0 \quad \forall u \in V \setminus \{s,t\}$$

Given :→ Given a flow-network $G = (V, E, c)$, source & sink vertices s & t

Find :→ Flow f that maximizes $\sum_{v \in V} f(s,v)$

Flow-Properties $\quad f(X,Y) = \sum_{x \in X, y \in Y} f(x,y)$, $\quad X, Y \subseteq V$

1) **Flow-conservation inside a vertexset** :

$$\forall X \subseteq V \quad f(X,X) = 0$$

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$$\hookrightarrow \sum_{\substack{x \in X \\ y \in X}} f(x, y) = \sum_{x \in X} \cancel{f(x, x)}^{\,0} + \sum_{\substack{x \in X \\ y \in X \\ x \neq y}} f(x, y)$$

$$\downarrow$$

$$\frac{1}{2} \sum_{\substack{x \in X \\ y \in X \\ x \neq y}} \Big(\underbrace{f(x, y) + f(y, x)}_{0} \Big)$$

2) Anti-symmetry between vertex sets

$$f(X, Y) = -f(Y, X)$$

$$= \sum_{\substack{x \in X \\ y \in Y}} f(x, y) = -\sum_{\substack{x \in X \\ y \in Y}} f(y, x) = -f(Y, X)$$

3) Additivity $\hookrightarrow$ $X, Y, Z \subseteq V, \quad X \cap Y = \phi$

$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$$

$$= \sum_{a \in X \cup Y} f(a, Z)$$

$$= \sum_{a \in X} f(a, Z) + \sum_{a \in Y} f(a, Z) = f(X, Z) + f(Y, Z)$$

$Y \subseteq X \qquad f(X \setminus Y, Z) = f(X, Z) - f(Y, Z)$

$f(t, V) = -f(s, V)$

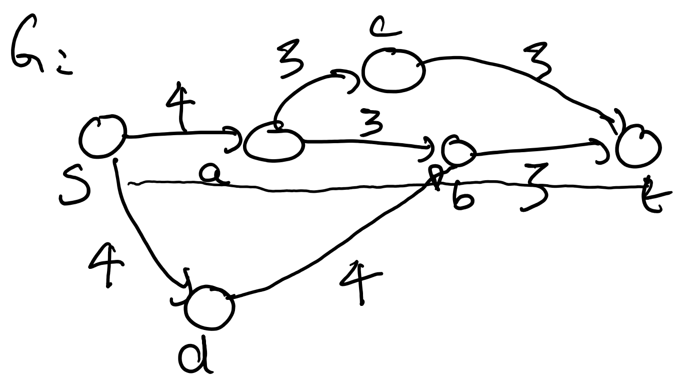
Residual network

Given a flow network $G = (V, E, c)$ & a flow f, the residual network $G_f = (V, E_f, c_f)$

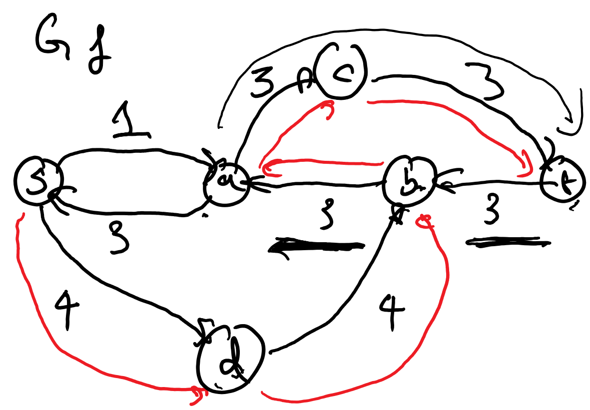
$$c_f(u,v) = c(u,v) - f(u,v) \quad \forall\, u,v$$

$$E_f = \{(u,v) \mid c_f(u,v) > 0\}$$

G:



G_f

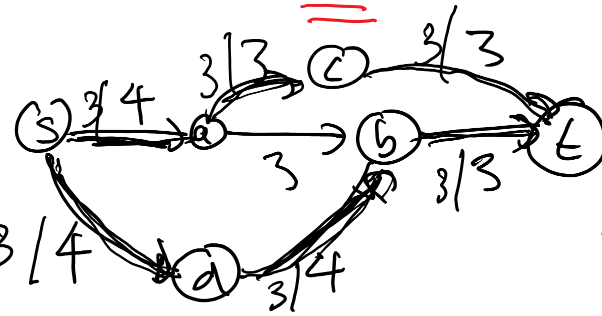


f = $s \xrightarrow{3/4} a \xrightarrow{3/3} b \xrightarrow{3/3} t$

f' = $s \to d \to b \to a \to c \to t$

3

$f + f'$ →



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$s \to a \to c \to t$

$s \to d \to b \to t$

Claim: If f' is a flow in G_f, then $f + f'$ is a flow in G

1) Capacity-constraint :→ $(f+f')(u,v)$

$$= f(u,v) + f'(u,v)$$
$$\leq f(u,v) + c(u,v) - f(u,v)$$
$$= c(u,v)$$

2) Anti-symmetry :→ $(f+f')(u,v)$

$$= f(u,v) + f'(u,v)$$
$$= -f(v,u) - f'(v,u)$$
$$= -(f+f')(v,u)$$

3) Flow conservation :→

$$\sum_{v\in V} (f+f')(u,v) \qquad \forall\, u \in V\setminus\{s,t\}$$

$$= \sum_{v\in V} \cancel{f(u,v)}^{\,0} + \sum_{v\in V} \cancel{f'(u,v)}^{\,0}$$

$$= 0$$

⇒ $f+f'$ is a flow in G.

$\boxed{\exists \text{ path from } s \text{ to } t \text{ in } G_f \Rightarrow \text{flow } f \text{ is not maximum.}}$

WEAK MAX-FLOW MIN CUT THM

max-flow in G ≤ min-cut in G

Claim :→ $f(s, V) = f(T, V\setminus T)$ $\quad S \in T$

$= f(T, V) - \cancel{f(T, T)}\ 0$ (by addition)

$= f(T \setminus \{s\} \cup \{s\}, V)$

$= f(s, V) + \overline{f(T\setminus\{s\}, V)}$

$= f(s, V)$ $\quad \cancel{\sum_{v \in T\setminus\{s\}} f(v, V)}\ 0$

Claim :→ $f(T, V\setminus T) \leq c(T, V\setminus T)$

$= \sum_{\substack{x \in T \\ y \in V\setminus T}} f(x, y) \leq \sum_{\substack{x \in T \\ y \in V\setminus T}} c(x, y)$

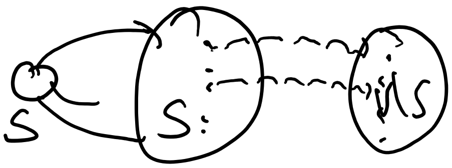
$= c(T, V\setminus T)$

$\Downarrow$

$$\boxed{f(s, V) \leq c(T, V\setminus T)}$$

$\nexists$ no path from s to t ⟹ f is a max-flow in G_f

S = set of vertices reachable from s in G_f



G_f

$\Rightarrow \exists$ no forward edges from S to $V \setminus S$ in G_f

$\Rightarrow f$ saturates all edges from S to $V \setminus S$

$f(u,v) = c(u,v)$

$\forall\ u \in S$
$v \in V \setminus S$.

$$\underbrace{f(S, V\setminus S)}_{\text{max-flow}} = \underbrace{c(S, V\setminus S)}_{\text{min-cut}}$$

Algorithm

while $\exists$ path from s to t in G_f
  augment flow along path. // $O(|E_f|)$
$\in O(|E|)$

$|F|$ = value of max-flow

$O(m|f|)$ (Ford-Fulkerson Algorithm)

MAX-FLOW-MIN-CUT-THEOREM

max-flow = min-cut

Edmond-Karp-Algorithm
(augments along shortest path from s to t)
→ $O(\underline{n}\,\underline{m}^2)$ algor
n: vertices in G
m: for max-flow
#edges

Jim-orlin → $O(nm)$ - time

Best running time using continuous optimization methods
$O\left(m^{3/2 - 1/328} \cdot \log c\right)$
c = max-capacity

$$f\underbrace{(s, V)}_{\text{max flow}} = f(S, V \setminus S) = c\underbrace{(S, V \setminus S)}_{\text{min-cut}}$$

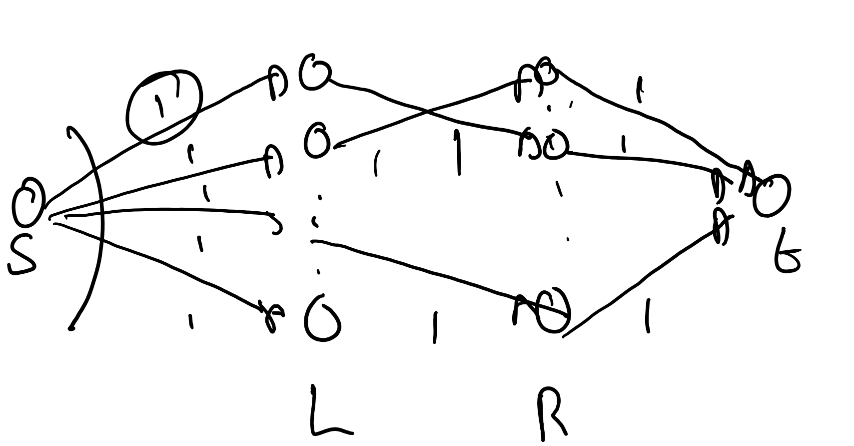
Weak MAX-FLOW MIN CUT THM

∀ flows f'
∀ cuts $S', V \setminus S'$

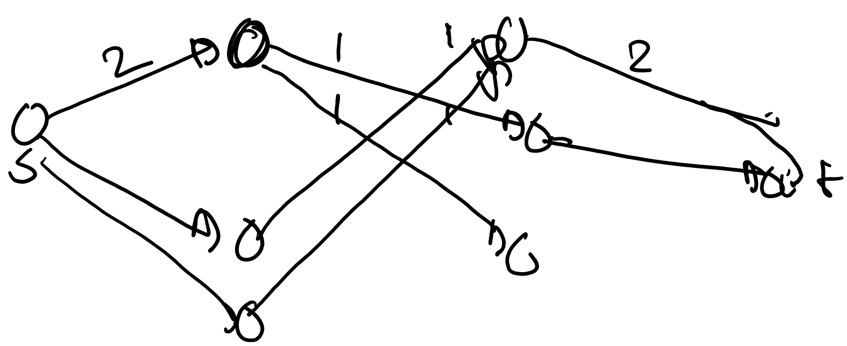
$$\underbrace{f'(s,V)}_{L_{max}} \leq c(s', V\setminus s')$$

Application

In $O(n,m)$ → find max matching in bipartite graphs



Find max-flow in the flow network



$O(nm)$

$O(\#\max \cdot m)$