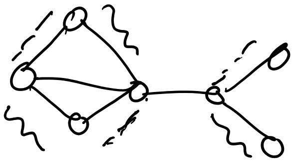
Matchings

4/11/2021

Given $G=(V,E)$. Matching $M \subseteq E$
s.t. all edges in M are vertex-disjoint



--- = form a matching

~~~ = form a matching

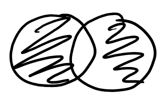
Given: Graph $G=(V,E)$
Find: Matching M of maximum size

Prelim: → consider a matching M
& a max-size matching M'

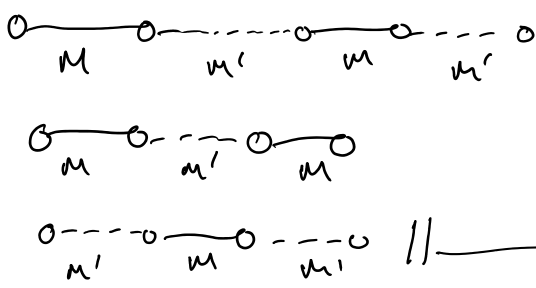
$M \neq M'$

↓

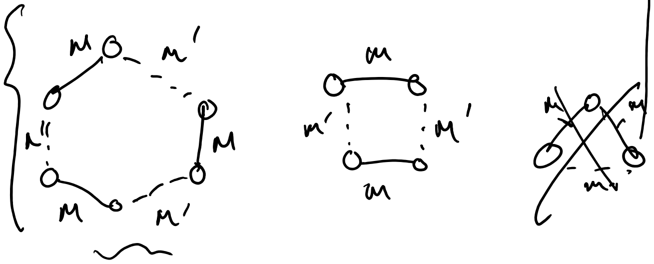
$$\underbrace{M \triangle M'}_{\text{symmetric difference}} = (M \setminus M') \cup (M' \setminus M)$$



→ set of "alternating paths"



→ set of "alternating cycles"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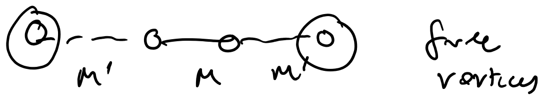
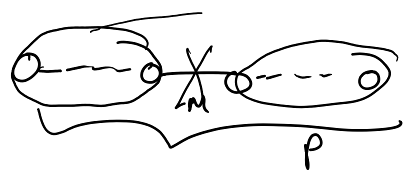
augmenting paths

↳ 

M is not a maximum matching $\Rightarrow$ $\exists$ augmenting path in G



$M'' \leftarrow M \triangle P$

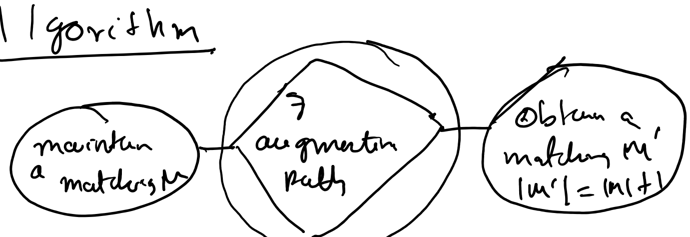
$\exists$ augmenting path in G $\Rightarrow$ M is not a max-size matching

M is a max-size matching

$\Updownarrow$

$\exists$ no augment path in G

Algorithm



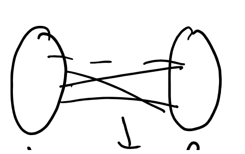
$T(m)$

$\Downarrow$

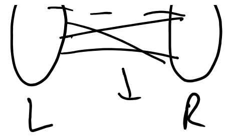
max-size matching in $O(nm)$

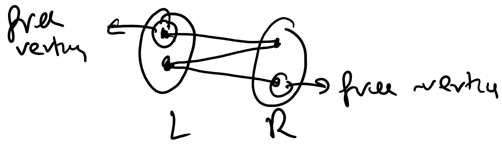
Bipartite Graphs

$G = (A \cup B, E)$

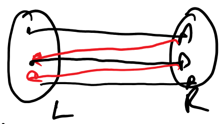


$G = (L \cup R, E)$

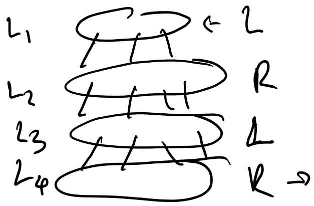




$G' \to$ direct edges in $E \setminus M$ from L to R



$\to$ direct edges in M from R to L



$O(n+m)$ to find an augmenting path in G

<u>Thm</u>: $\to$ max-size matching in bipartite graphs can be found in $O(nm)$ time

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<u>STABLE MATCHINGS</u>

<u>Given</u>: $\to$

set $M = \{m_1, m_2, \dots, m_n\}$ of n men

set $W = \{w_1, w_2, \dots, w_n\}$ of n women

each man m_i has a preference order over W

$$w_2 >_{m_i} w_3 >_{m_i} w_1 >_{m_i} w_5 > \dots >_{m_i} w_k$$

each woman w_j has a preference order over M

$$m_5 >_{w_j} m_1 >_{w_j} m_8 >_{w_j} \dots >_{w_j} m_k$$

blocking-pair → Given a matching M, (m_i, w_j) form a blocking pair iff both m_i & w_j prefer each other over their matched partners

stable-matching → Matching M, s.t. there are no blocking pairs

Goal:→ Find a stable matching M

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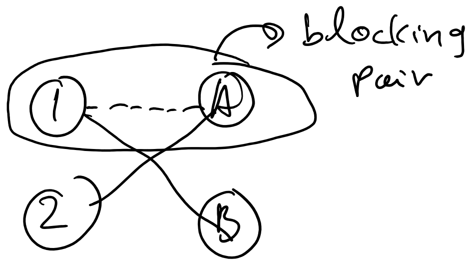
1 A

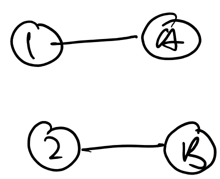
2 B

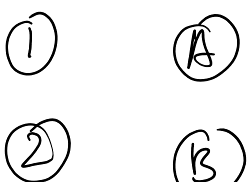
MEN WOMEN

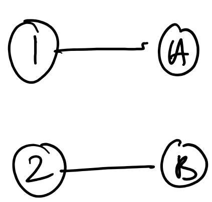
$A \succ_{1,2} B$

$1 \succ_{A,B} 2$

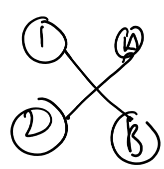








"man-biased" stable matching



"women biased" stable matching

Preferences

men	women
$A \succ B$	$2 \succ 1$

men	women	matching	matching
$A \succ_1 B$	$2 \succ_A 1$		
$B \succ_2 A$	$1 \succ_B 2$		

Gale-Shapely Algorithm (GS)

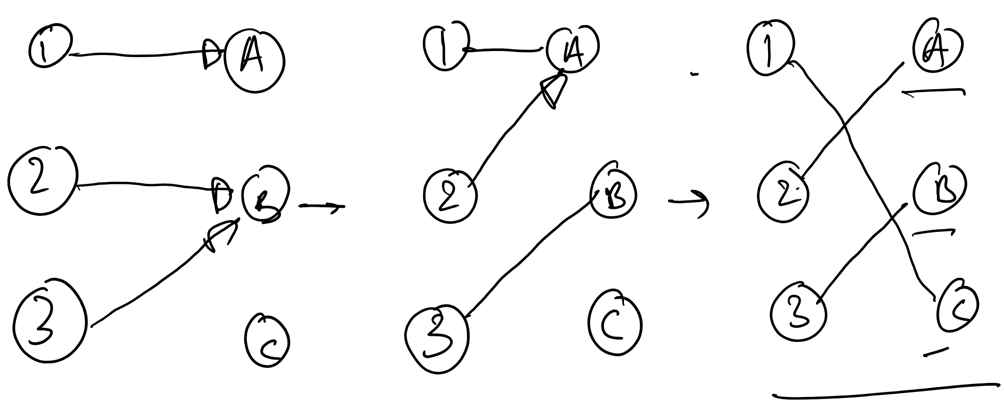
while $\exists$ unmatched man,

each unmatched man m_i proposes their best woman w_j, whom they have not proposed before

if w_j prefers m_i over her current matched partner

match (m_i, w_j)

unmatch w_j with her previous partner



Round 1

STABLE

Preferences

men	women	
$A \succ C \succ B$	$2 \succ 1 \succ 3$	

men	women
$A >_1 C >_1 B$	$2 >_A 1 >_A 3$
$B >_2 A >_2 C$	$3 >_B 2 >_B 1$
$B >_3 A >_3 C$	$1 >_C 2 >_C 3$

Correctness → outputs a stable matching

Claim: → every man is matched.

Assume otherwise

⇓

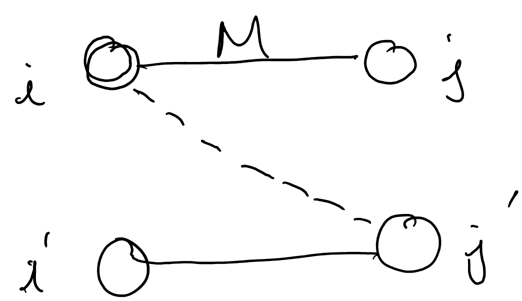
∃ man rejected by all women

⇓

all women are matched. ⚡

why stable? M is the output of GS

Assume otherwise



(i, j') = blocking pair

$(i, j') =$ blocking pair

$j' >_i j$

$i >_{j'} i \Rightarrow (i, j')$ is not a blocking pair $\lightning$

Running time $\Rightarrow$ $O(\#\text{proposals made})$

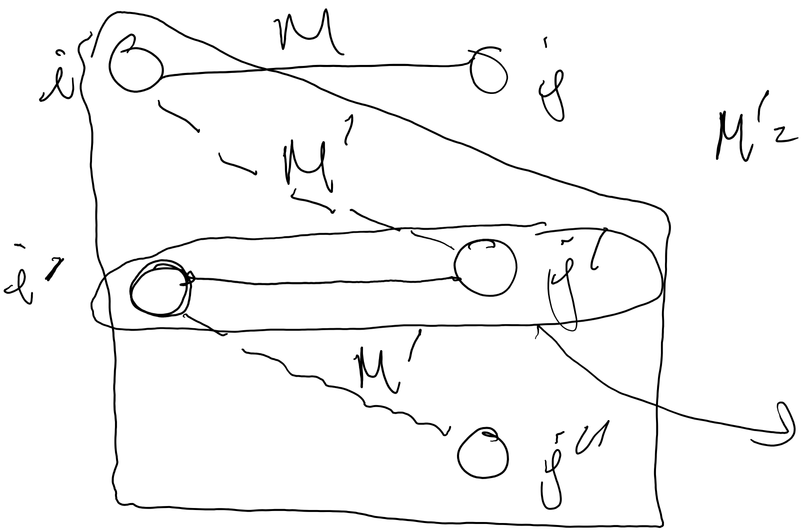
$= O(\ \ )$

Thm $\Rightarrow$ At the end of GS, every man is matched to the best women he can be matched in any stable matching.

Stable pair $\Rightarrow$ (w, j) form a stable pair if $\exists$ a stable

Matching where i & j are matched

Execution $\mathcal{E}$ of the GS algorithm



M' = another Stable Matching

$j' \succ_i j$

$\boxed{i' \succ_{j'} i}$

Rejection (i, j') = rejection of a stable pair

First time w.l.o.g. in $\mathcal{E}$, that a rejection of a stable pair has happened

$j' \succ_{w'} j''$ $\qquad$ $i' \succ_{j'} i$

ALWAYS PROPOSE