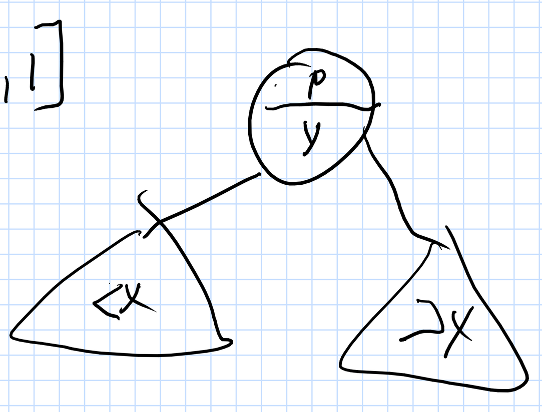
L13: Treaps, Chernoff's inequality, Min cut

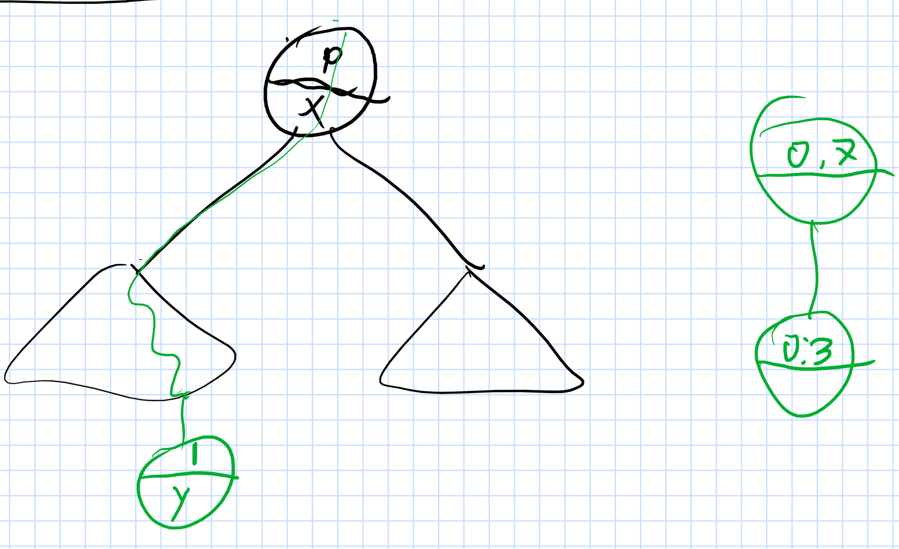
10/7/21

$X_1, X_2, \dots X_n$

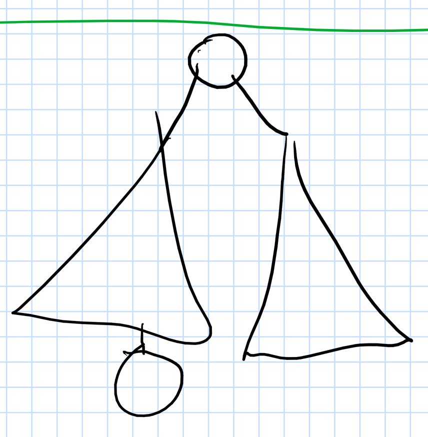
$P_1, P_2, \dots, P_n \in [0,1]$

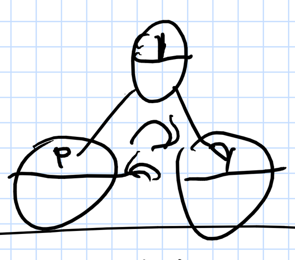


y, q

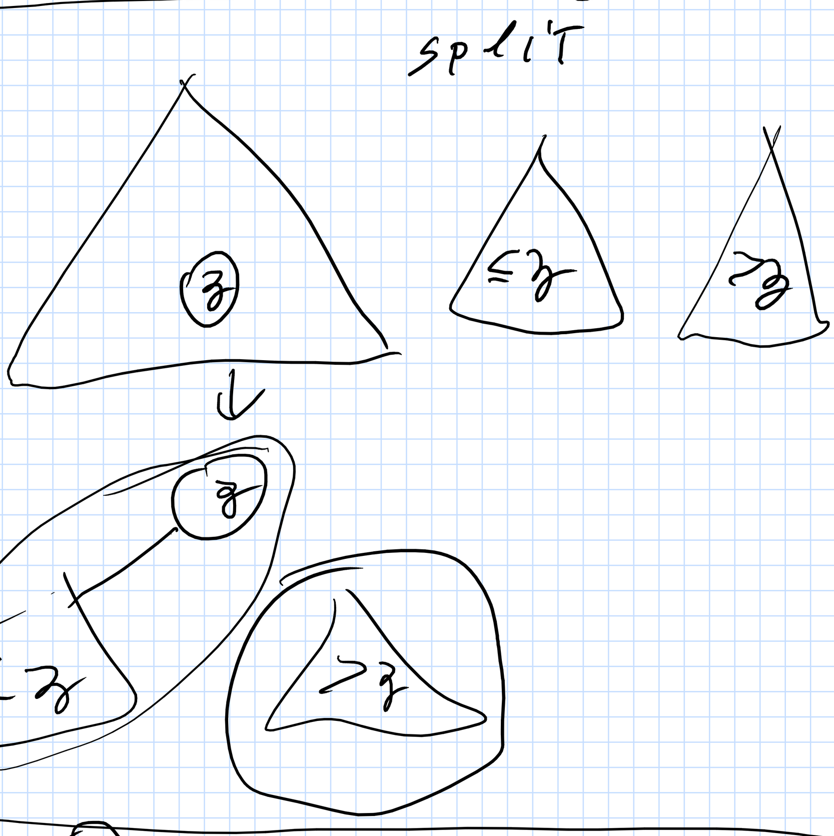


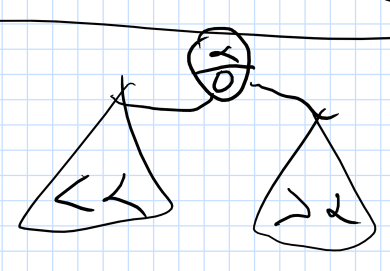
Deletion



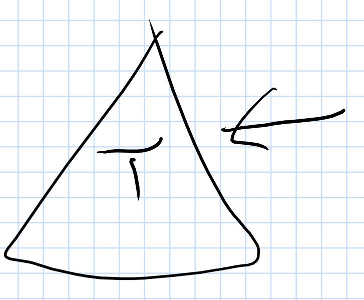


split





meld



$\leftarrow X_1, \dots, X_n$

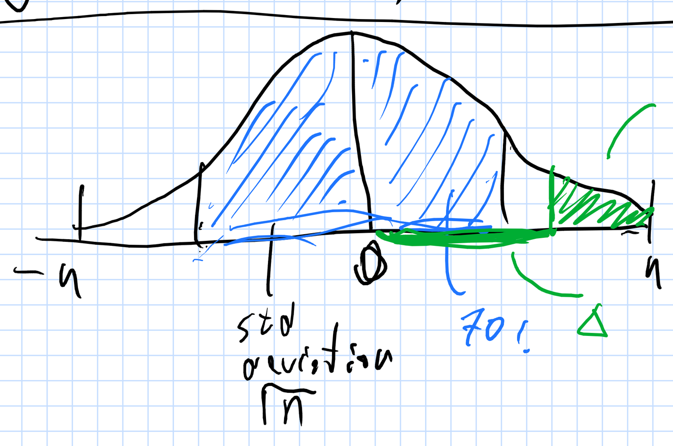
Chernoff inequality

$X_1, \dots, X_n \in \{-1, +1\}$ with eq probability independently.

$$Y = \sum_{i=1}^{n} X_i$$

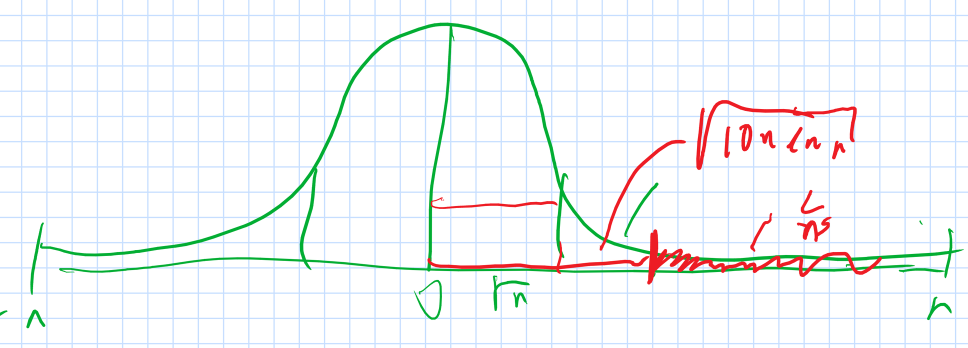
$$P[Y > \Delta] \le \exp\left(-\frac{\Delta^2}{2n}\right).$$

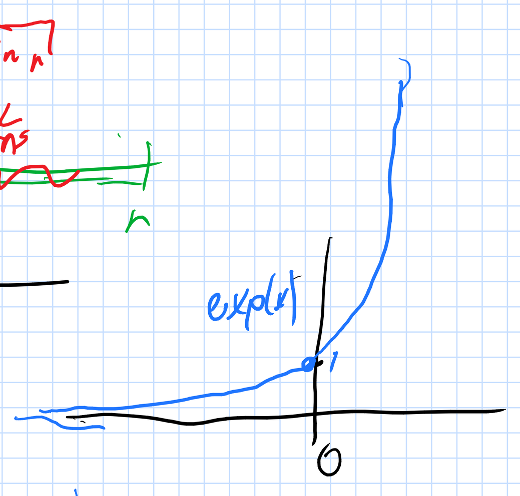
Y



$$\le \exp\left(-\frac{\Delta^2}{2n}\right) = \exp\left(-\frac{10\, n \ln n}{2n}\right) = \exp(-5 \ln n) = \frac{1}{n^5}$$

$\Delta = \sqrt{10\, n \ln n}$

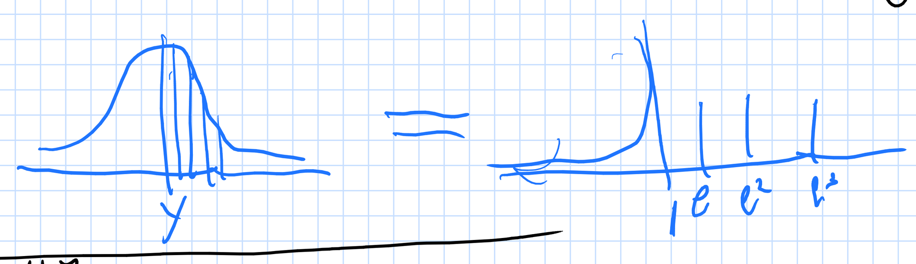




Proof $t > 0$

$$P[Y>\Delta] = P[tY > t\Delta] = P[\exp(tY) > \exp(t\Delta)]$$

$Y \Rightarrow e^Y$



$$P[e^{tY} > e^{t\Delta}] \le \frac{E[e^{tY}]}{\exp(t\Delta)} = \qquad e^{a+b} = e^a e^b$$

$Y=1$
$Y=2$

$$E[e^{tY}] = E\left[\exp\left(\sum_{i=1}^{n} tX_i\right)\right]$$
$$= E[e^{tX_1} e^{tX_2} \cdots e^{tX_n}] \qquad E[X \cdot Y] = E[X]E[Y]$$
$$= \prod_{i=1}^{n} E[e^{tX_i}] \le$$
$$= \left(E[\exp(tX_1)]\right)^n \le \exp\left(\frac{t^2 n}{2}\right)$$

$$E[\exp(tX_1)] = \underbrace{\tfrac{1}{2}\exp(t)}_{X_1=1} + \underbrace{\tfrac{1}{2}\exp(-t)}_{X_1=-1}$$
$$\le \frac{1}{2} 2 \exp\left(\frac{t^2}{2}\right)$$
$$\le \exp\left(\frac{t^2}{2}\right)$$

Taylor expansion

$$\exp(t) = 1 + \cancel{\frac{t}{1!}} + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$
$$+$$
$$\exp(-t) = 1 - \cancel{\frac{t}{1}} + \frac{t^2}{2!} - \frac{t^3}{3!} + $$
$$= 2\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \dots\right)$$

$$\boxed{\frac{t^{2k}}{(2k)!} \le \frac{t^{2k}}{2^k k!} = \frac{1}{k!}\left(\frac{t^2}{2}\right)^k \qquad (2k)! \ge 2^k k!}$$

$$\le 2\left(1 + \frac{1}{1!}\left(\frac{t^2}{2}\right) + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \dots\right)$$
$$= 2\exp\left(\frac{t^2}{2}\right)$$

$$P[Y > \Delta] \le \frac{E[\exp(tY)]}{e^{t\Delta}} \le \frac{\exp(nt^2/2)}{\exp(t\Delta)}$$
$$= \exp\left(\frac{nt^2}{2} - t\Delta\right) \qquad \boxed{t = \frac{\Delta}{n}}$$
$$= \exp\left(\frac{n}{2}\frac{\Delta^2}{n^2} - \frac{\Delta^2}{n}\right)$$
$$= \exp\left(\frac{\Delta^2}{2n} - \frac{\Delta^2}{n}\right)$$
$$= \exp\left(-\frac{\Delta^2}{2n}\right).$$

∎

Galton-Watson processes

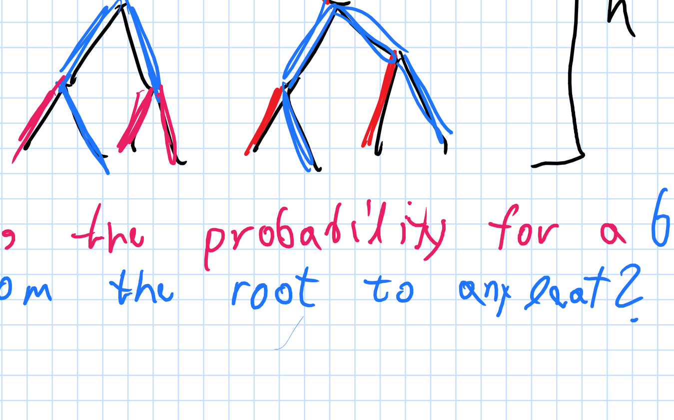
$X = 0, 1, 2, \dots$

random distribution

$E[X] > 1 \qquad P[\text{Tree be infinite}] > 0$

$E[X] < 1 \qquad P[\text{Tree is infinite}] = 0$

$E[X] = 1 \qquad P[\text{Tree is infinite}] = 0$



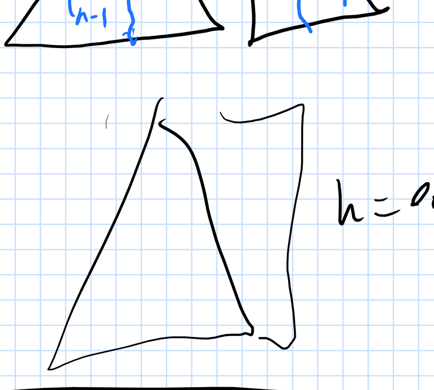
Q: What is the probability for a blue path from the root to any leaf?

P_h

$P_0 = 1 \qquad P_1 = \frac{3}{4}$

$$P_h = 1 - \left(1 - \frac{P_{h-1}}{2}\right)^2$$

$$P_h = P_{h-1} - \frac{P_{h-1}^2}{4}$$



Lemma $P_h \ge \frac{1}{h+1}$.
$$\ge \frac{1}{\log n}$$