473 Lecture 12 10/5/21

More randomized algorithms

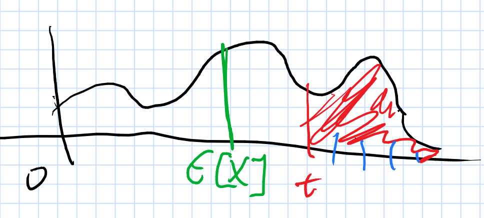
- Markov inequality
- Max cut
- Conditional expectation
- QS High prob
- Treaps

Markov's inequality

X : Random variable

$X \geq 0$

$E[X]$ = expectation of X $= \sum_x x\, P[X=x]$.



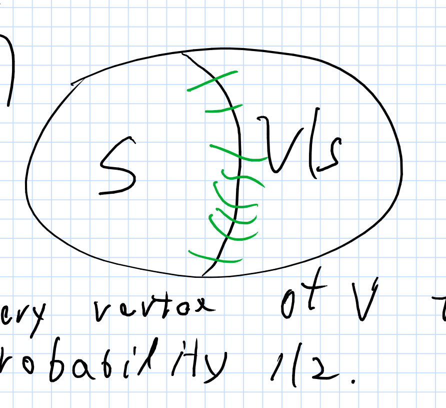
$P[X \geq t] \leq \frac{E[X]}{t}$

Proof $P[X \geq t] > \frac{E[X]}{t} \Rightarrow E[X] \geq t \cdot P[X \geq t] > t \cdot \frac{E[X]}{t} = E[X]$.

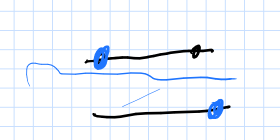
Max Cut

$G = (V, E)$

NP Hard



Pick every vertex of V to be in S with probability 1/2.



$P[\text{an edge to be in the cut}] = \frac{1}{2}$

$e_1, e_2, \dots, e_m$ be the edges of G

$X_i = 1 \iff e_i \in (S, V \setminus S)$

$E[X_i] = P[X_i = 1] = \frac{1}{2}$

$E[\text{size of the cut}] = E\left[\sum_{i=1}^m X_i\right] = \sum_{i=1}^m E[X_i] = \sum_{i=1}^m \frac{1}{2} = \frac{m}{2}$.

How to get a cut with at least $(\frac{1}{2} - \delta)m$ for $\delta \in (0, \frac{1}{2})$.

Z = # of edges not in the cut

$E[Z] = \frac{m}{2}$

$P[\#\text{ edges in the cut} \leq (\frac{1}{2} - \delta)m] = P[Z \geq (\frac{1}{2} + \delta)m]$
$= P[Z \geq (1+2\delta)\frac{m}{2}]$
$= P[Z \geq (1+2\delta)E[Z]] \leq \frac{E[Z]}{t} = \frac{1}{1+2\delta} \leq 1 - \delta$ (Markov's ineq)

$P[\#\text{ edges in cut} \leq (\frac{1}{2} - \delta)m] \leq 1 - \delta$

Run alg t times. $\delta = 0.1$

Then

$P[\text{all executions fail}] = P[F_1 \cap F_2 \cap \dots \cap F_t] = \prod_{i=1}^t P[F_i]$

$P[A \cap B] = P[A]\,P[B]$ $\leq (1-\delta)^t \leq 0.01$

A and B are indep. Amplification

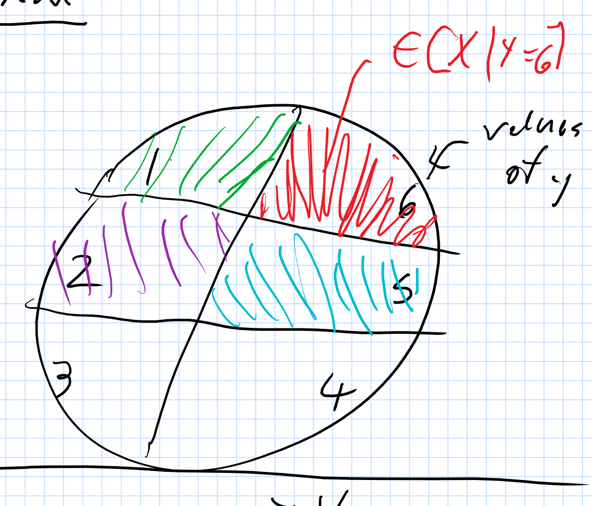
$(1-\delta)^t \leq \exp(-\delta t)$

$t = \frac{1}{\delta}$

Conditional expectation

$E[X|Y] = E[X|Y=y]$ $= f(y)$

$E_y[E[X|Y=y]]$ $= E[X]$.

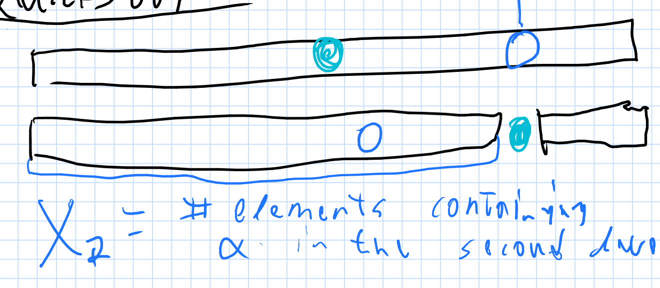


$X_1 \to X_2 \to X_3 \to \dots \to X_n$

$E[X_i | X_{i-1}]$

$E[X_i] = E[E[X_i | X_{i-1}]]$

QuickSort



$X_1 = n$

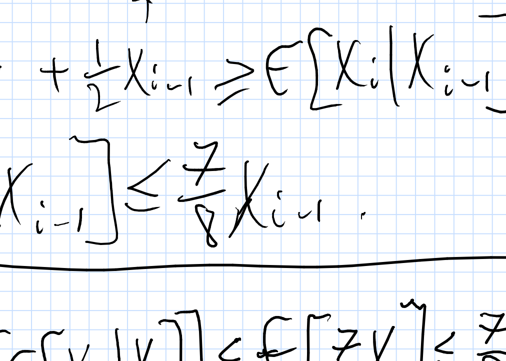
X_2

X_2 = # elements containing α in the second level

X_i = # elements in the subproblem containing α in the ith level of recursion of QuickSort.

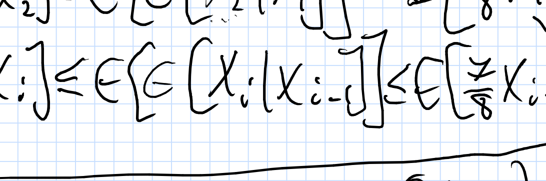
$X_1, X_2, X_3, \dots X_t = 1$ How big can t be

$E[X_i | X_{i-1}]$



claim $E[X_i | X_{i-1}] \leq \frac{7}{8} X_{i-1}$.

proof



$\frac{1}{2}\,\frac{3}{4}X_{i-1} + \frac{1}{2}X_{i-1} \geq E[X_i | X_{i-1}]$

$E[X_i | X_{i-1}] \leq \frac{7}{8} X_{i-1}$.

$X_1 = n$

$E[X_2] = E[E[X_2|X_1]] \leq E[\frac{7}{8}X_1] \leq \frac{7}{8}n$

$E[X_i] \leq E[E[X_i | X_{i-1}]] \leq E[\frac{7}{8}X_{i-1}] = \frac{7}{8}E[X_{i-1}]$ $\leq (\frac{7}{8})^i n$.

$m = 10 \log_{8/7} n = 10 \frac{\ln n}{\ln 8/7} = O(\log n)$

$E[X_m] \leq (\frac{7}{8})^m = \exp(m \ln \frac{7}{8})$
$= \exp(-m \ln \frac{8}{7})$
$= \exp(-10 \frac{\ln n}{\ln \frac{8}{7}} \cdot \ln \frac{8}{7})$
$= \frac{1}{n^{10}}$.

$P\left[\begin{matrix}\alpha \text{ participates} \\ \text{in more than} \\ m \text{ recursive calls}\end{matrix}\right] \leq P[X_m \geq 1] \leq \frac{E[X_m]}{1} = E[X_m] \leq \frac{1}{n^{10}}$.

$E[X_m] \leq \frac{1}{n^{10}}$

$\mathcal{E}_i$ = the ith element of the input participates in more than m recursive calls of QS.

$P[\bigcup_{i=1}^n \mathcal{E}_i] \leq \sum_{i=1}^n P[\mathcal{E}_i] \leq \sum_{i=1}^n \frac{1}{n^{10}} = \frac{1}{n^9}$.

↑ union bound.

$\Rightarrow$ with prob $\geq 1 - \frac{1}{n^9}$ QS performs at most mn comparisons

$mn \approx 20 n \log n$.

$\Rightarrow$ QS runs in $O(n \log n)$ time with high probability

$\text{Prob} \xrightarrow[n \to \infty]{} 1$ $\quad P \geq 1 - \frac{1}{n^{O(1)}}$

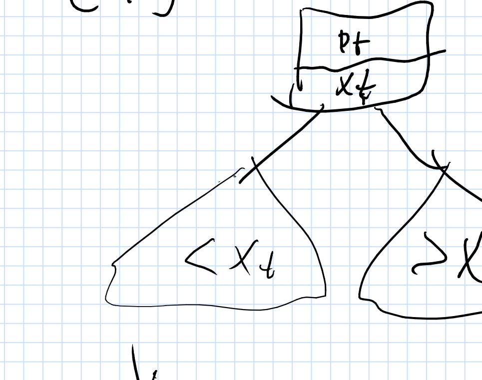
Let M be the number of times we flip a coin. Let Y be the number of times we got heads in these M rounds.
We have that
$E[Y] = \frac{M}{2}$

$P[Y \leq \frac{M}{4}] \leq 2^{-M/16}$

Treaps

red-black trees
AVL trees

$x_1, x_2, \dots x_n$



$x_1, x_2, \dots x_n$
$p_1, p_2, \dots p_n \in [0,1]$

Heap on the priorities