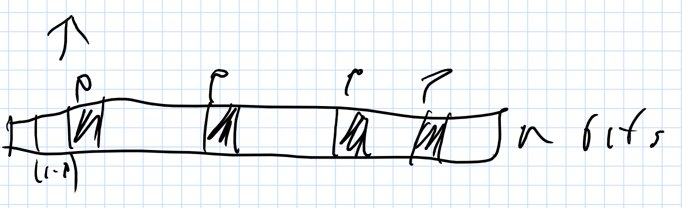
L11: Randomized algorithms

$X \sim Bin(n,p)$ n = # of coin flips, p = probability for heads

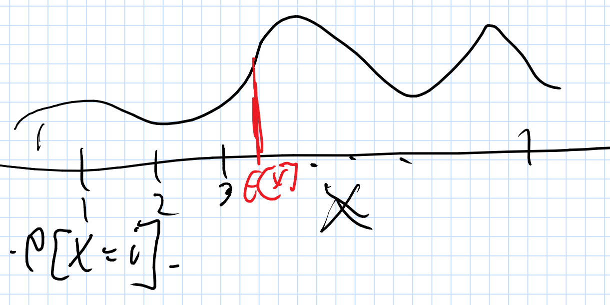
$P[X=i] = \binom{n}{i} p^i (1-p)^{n-i}$



$\binom{n}{i} = \frac{n!}{i!(n-i)!}$

$X \sim Bin(500, \frac{1}{2})$

$P[X=i]$ = prob of X got value i.



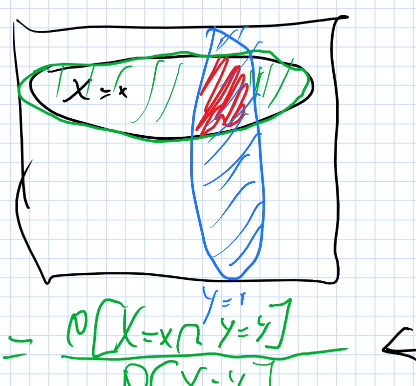
$E[X] = \sum i \cdot P[X=i]$

expectation

$E[X+Y] = E[X] + E[Y]$ linearity of expectation

Conditional probability

$P[X=x \mid Y=y] \equiv$ probability of $X=x$ given that $Y=y$.



$P[X=x \mid Y=y] = \frac{P[X=x \wedge Y=y]}{P[Y=y]}$

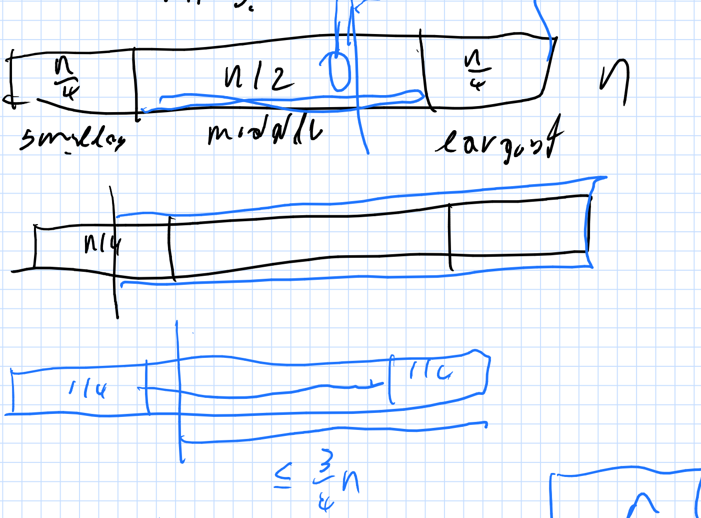
X and Y are independent if $P[X=x \mid Y=y] = P[X=x]$

If X and Y are independent then $E[XY] = E[X]E[Y]$

Nuts and bolts

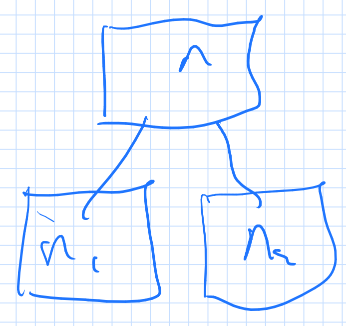
Do quicksort!

$T(n)$: Expected running time of QS for n numbers.



n_1, n_2: sizes of the

- Lucky with prob $\frac{1}{2}$: $n_1, n_2 \le \frac{3}{4}n$
- Unluck

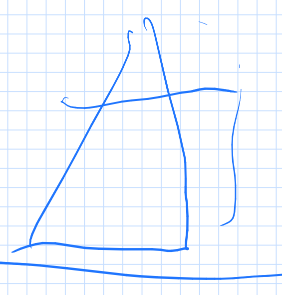


$T(n) \le \frac{1}{2} T(n) + \frac{1}{2}\left(T\left(\frac{n}{4}\right) + T\left(\frac{3}{4}n\right)\right) + O(n)$ (unlucky)

$\frac{1}{2} T(n) \le \frac{1}{2}\left(T\left(\frac{n}{4}\right) + T\left(\frac{3}{4}n\right)\right) + O(n)$

$T(n) \le T\left(\frac{n}{4}\right) + T\left(\frac{3}{4}n\right) + O(n)$

$= O(n \log n)$



Quick sort Analysis

Indicator variable $X \in \{0,1\}$

$X=1 \iff$ a certain event happened.

$X_{i,j} = 1 \iff$ QS compared the ith smallest element to the jth smallest element. ($i<j$, $\binom{n}{2}$)

RT of QS $= O(\sum X_{i,j})$

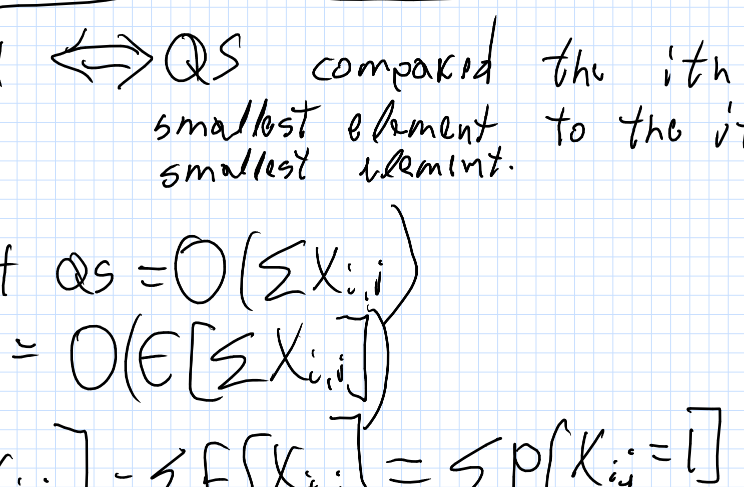
$E[RT] = O(E[\sum X_{i,j}])$

$E[\sum X_{i,j}] = \sum E[X_{i,j}] = \sum P[X_{i,j}=1]$

$E[X_{i,j}] = P[X_{i,j}=0] \cdot 0 + P[X_{i,j}=1] \cdot 1 = P[X_{i,j}=1]$

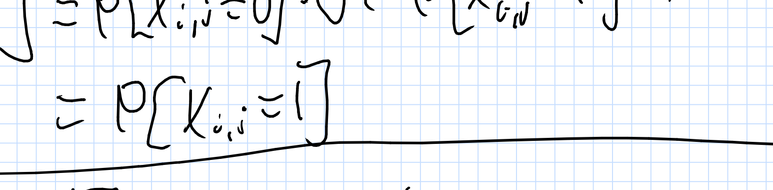
$P[X_{i,j}=1] = 1$ $j = i+1$

$j = i + \frac{n}{4}$



$p > \alpha_i$ and α_j

$\alpha_i < p < \alpha_j \Rightarrow \alpha_i$ and α_j get separated $\Rightarrow X_{i,j} = 0$



Run quicksort till a pivot is chosen from the set $\alpha_i, \alpha_{i+1}, \ldots, \alpha_j$ $\alpha[i,j]$

$P[X_{i,j}=1] = P\left[\text{pivot} = \alpha_i \text{ or pivot} = \alpha_j \mid \text{this is the first pivot in } \alpha[i,j]\right]$

$= \frac{2}{|\alpha[i,j]|} = \frac{2}{j-i+1}$

$\sum_{i<j} E[X_{i,j}] = \sum_{i<j} P[X_{i,j}]$

$= \sum_{i<j} \frac{2}{j-i+1}$

$\le \sum_{i=0}^{n} \sum_{\Delta=1}^{n} \frac{2}{j-i+1} = \sum_{i=1}^{n} \sum_{\Delta=1}^{n} \frac{2}{\Delta+1}$ ($j = i+\Delta$)

$= 2n \cdot \sum_{\Delta=1}^{n} \frac{1}{\Delta+1}$

$\le 2n \left(\sum_{i=1}^{n} \frac{1}{i}\right)$

$= 2n H_n = O(n \log n)$

$H_n = \ln n + O(1)$