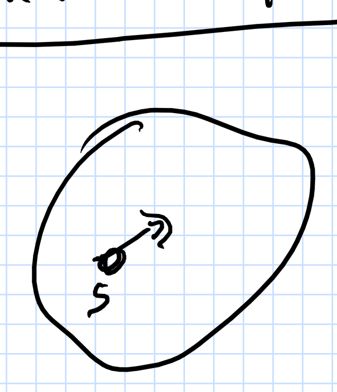
473 Lecture 6: More dynamic programming
9/9/2021

## Bellman-Ford algorithm

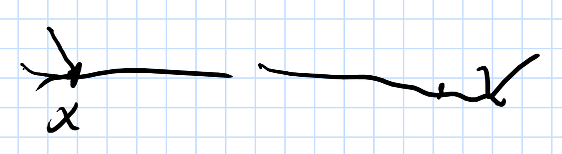


$G$, weights on the edges

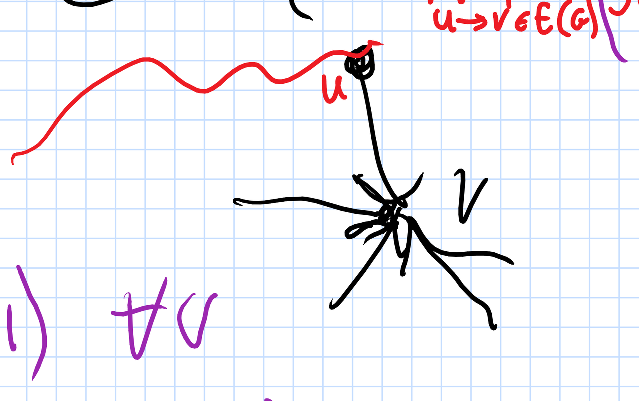
Task: Compute shortest path from $s$ to all vertices in the graph.

weights can be negative.

$g(v,i) \equiv$ length of shortest ~~path~~ walk in $G$ from $s$ to $v$ using at most $i$ edges

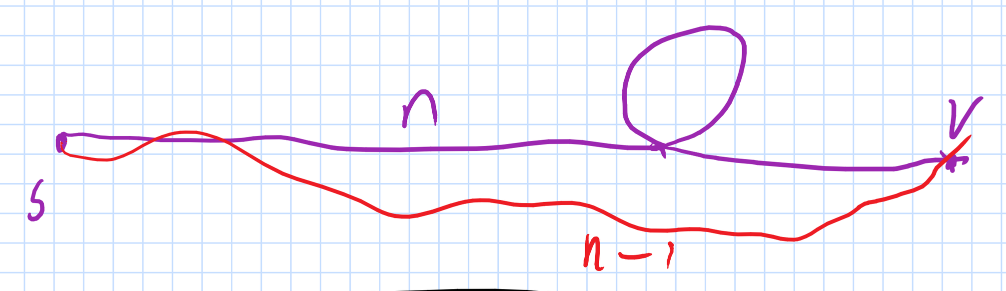


$$g(v,0) = \begin{cases} 0 & v = s \quad i=0 \\ \infty & v \neq s \quad i=0 \\ \min\left( g(v,i-1),\ \min_{u \to v \in E(G)} g(u,i-1) + \ell(u \to v) \right) & i > 0 \end{cases}$$



$g(v,i-1)\ \forall v$

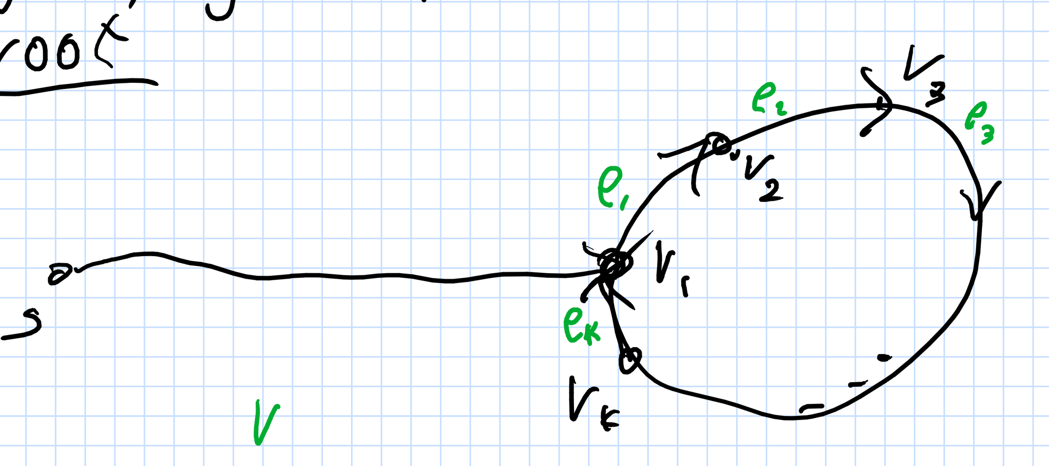
$\Rightarrow g(u,i)\ \forall u \quad O(m)$



**claim**

If there is a negative cycle in $G$ reachable from $s$ then $\exists v \in V(G)$ s.t. $g(v,n) < g(v,n-1)$.

**proof**



$g(v_i,n) \le g(v_{i-1},n-1) + \ell(e_{i-1})$ | $v_0 = v_k$, $e_0 = e_k$

$\sum_{i=1}^{k} g(v_i,n) \le \sum_{i=1}^{k} g(v_{i-1},n-1) + \sum_{i=1}^{k} \ell(e_i)$

$\ell(C) < 0$

$\sum_{i=1}^{k} \left( g(v_i,n) - g(v_i,n-1) \right) \le \ell(C) < 0$

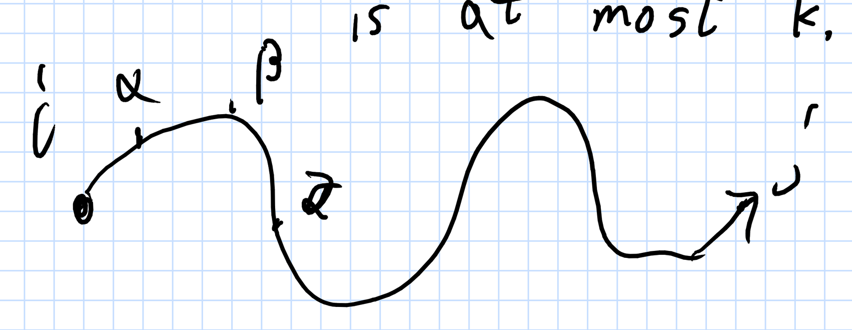
$g(v_i,n) - g(v_i,n-1) < 0$

## Floyd Warshall

$G = (V,E) \quad V = [n] = \{1,2,\ldots,n\}$

weights on the edges (can be negative).

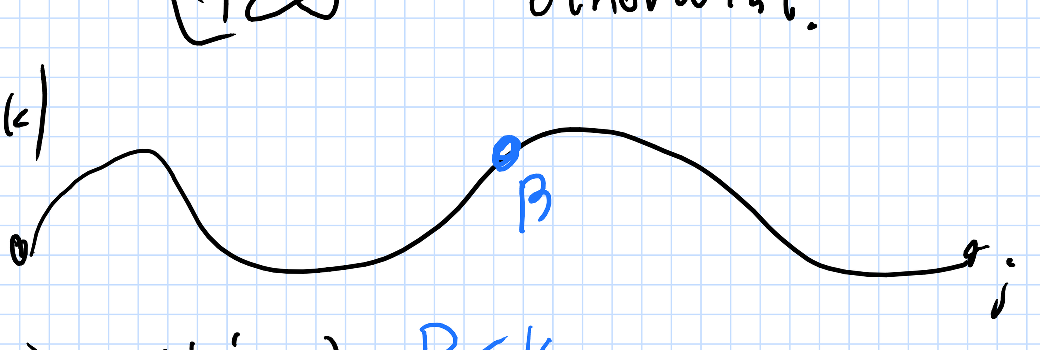
$g(i,j,k) =$ length of shortest path from $i$ to $j$ s.t. the highest vertex on the path is at most $k$.



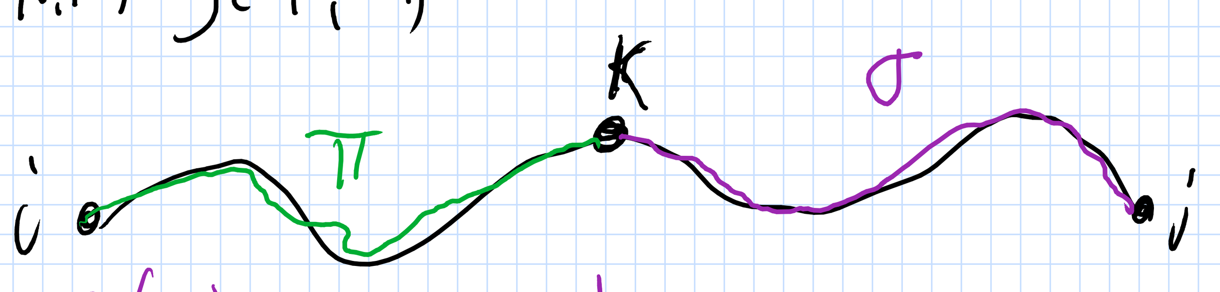
$\max(\alpha,\beta,\gamma,\ldots) \le k$

$$g(i,j,0) = \begin{cases} \ell(i \to j) & i \to j \in E(G) \\ +\infty & \text{otherwise} \end{cases}$$

$g(i,j,k)$



$g(i,j,k) = g(i,j,k-1) \quad \beta < k$



$\ell(\pi) = g(i,k,k-1)$

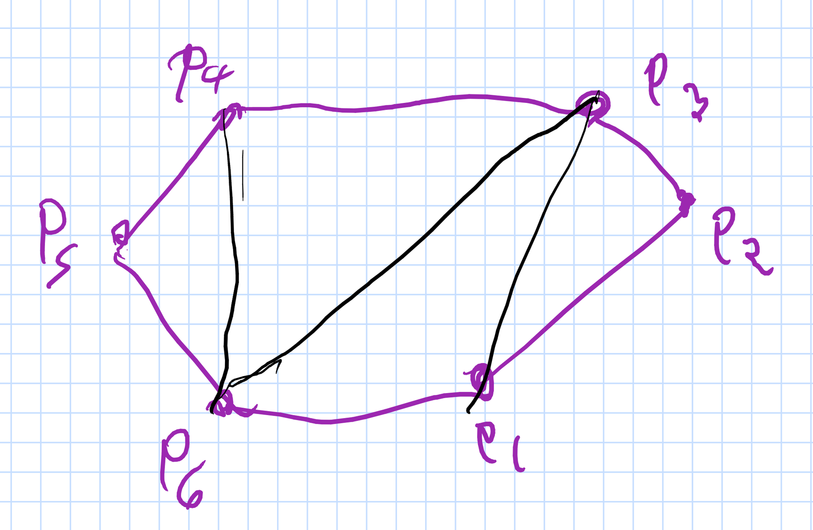
$\ell(\sigma) = g(k,j,k-1)$

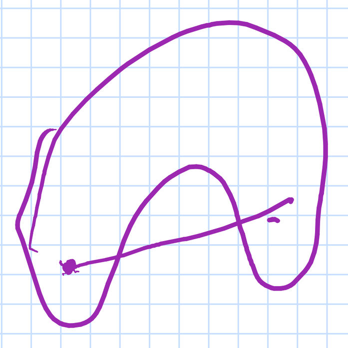
$$g(i,j,k) = \min\begin{pmatrix} g(i,j,k-1) \\ g(i,k,k-1) + g(k,j,k-1) \end{pmatrix} \quad k > 0$$

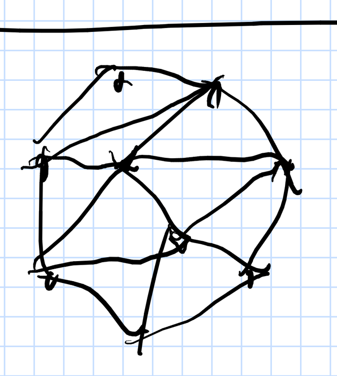
for $i,j$ we want to compute $g(i,j,n-1)$.

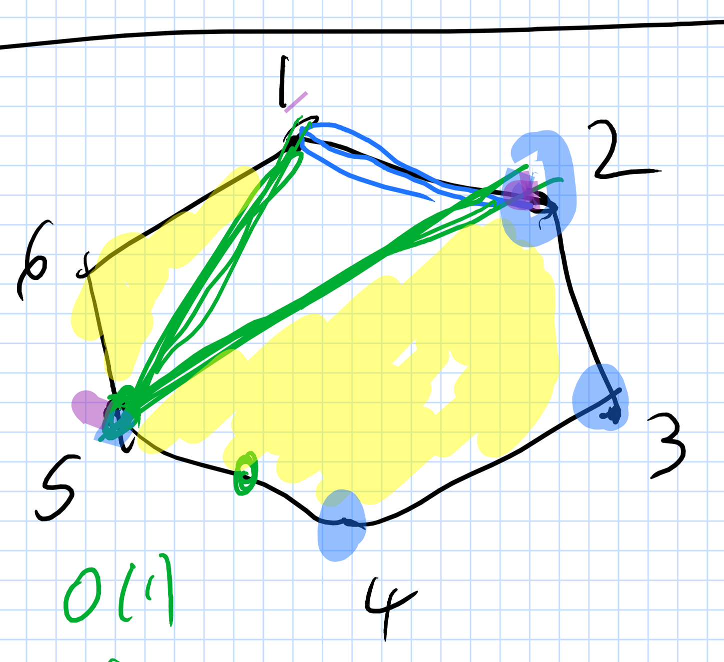
**Thm** Floyd-Warshall computes APSP in $O(n^3)$ time.

$p_1, p_2, p_3, \ldots, p_n$ convex position









$O(1)$

$$f(i,j) = \begin{cases} |s_i s_{i+1}| & j = i+1 \\ \min_k |s_i s_k| + f(i,k) + f(k,j) \end{cases}$$

$O(n^2)$    $O(n)$

$O(n^3)$ time.