Problem

1. Prove that if $X$ is a finite set and $Y$ is a subset of $X$, then $|Y| \leq |X|$. Draw a diagram to illustrate the problem.

Solution

- Use induction on the size of $X$.
- Base case: If $|X| = 1$, then $Y$ can only be $\emptyset$ or $\{1\}$, so $|Y| \leq 1 = |X|$.
- Inductive step: Assume the statement is true for all sets $Z$ with $|Z| < n$. Let $X$ be a set with $|X| = n$. If $Y = X$, then $|Y| = n = |X|$. Otherwise, remove an element $x \in X$ to get a set $Z$ with $|Z| = n - 1$. By the inductive hypothesis, $|Y_{Z}| \leq |Z| = n - 1$. So $|Y| = |Y_{Z}| + 1 \leq |Z| + 1 = n = |X|$.

Diagram:

- Draw a Venn diagram showing $X$ and $Y$.
- Shade $Y$ within $X$.
- Illustrate the inductive step by showing $Z$ and $Y_{Z}$.

2. Let $X$ and $Y$ be sets. Define a relation $R$ on $X$ by $xRy$ if and only if $x$ is an ancestor of $y$ in a directed tree $T$. Prove that $R$ is an equivalence relation.

Solution

- Reflexive: For any $x \in X$, $x$ is an ancestor of itself in $T$, so $xRx$.
- Symmetric: If $xRy$, then $x$ is an ancestor of $y$ in $T$. Since parent-child relationships are symmetric, $y$ is an ancestor of $x$ in $T$, so $yRx$.
- Transitive: If $xRy$ and $yRz$, then $x$ is an ancestor of $y$ and $y$ is an ancestor of $z$ in $T$. Therefore, $x$ is an ancestor of $z$ in $T$, so $xRz$.

Diagram:

- Draw a directed tree $T$.
- Label $x$, $y$, and $z$.
- Indicate the parent-child relationships.

3. Let $X$ be a set and $R$ be a relation on $X$. Define a function $f: X \to \mathcal{P}(X)$ by $f(x) = \{y \in X : yRx\}$. Prove that $f$ is one-to-one if and only if $R$ is a partial order.

Solution

- If $f$ is one-to-one, then for any $x, y \in X$, $f(x) = f(y)$ implies $x = y$. This means that $yRx$ implies $x = y$, so $R$ is a partial order.
- If $R$ is a partial order, then for any $x, y \in X$, $yRx$ implies $x = y$, so $f(x) = \{y\}$. This means that $f(x) = f(y)$ implies $x = y$, so $f$ is one-to-one.

Diagram:

- Draw a Venn diagram showing $X$.
- Label $x$ and $y$.
- Indicate the parent-child relationship.

4. Let $X$ be a set and $R$ be an equivalence relation on $X$. Define a function $g: X \to \mathcal{P}(X)$ by $g(x) = \{y \in X : xRy\}$. Prove that $g$ is surjective.

Solution

- For any $Y \subseteq X$, let $x \in X$. If $x \in Y$, then $g(x) = Y$. If $x \notin Y$, then $g(x)$ is a singleton set containing $x$, which is contained in $Y$. Therefore, $g(x) \subseteq Y$.

Diagram:

- Draw a Venn diagram showing $X$.
- Label $x$ and $Y$.
- Indicate the containment relationship.

5. Let $X$ be a set and $R$ be an equivalence relation on $X$. Define a function $h: X \to \mathcal{P}(X)$ by $h(x) = \{y \in X : xRy\}$. Prove that $h$ is injective.

Solution

- For any $x, y \in X$, if $h(x) = h(y)$, then $xRy$ and $yRx$. Since $R$ is an equivalence relation, $x = y$.

Diagram:

- Draw a Venn diagram showing $X$.
- Label $x$ and $y$.
- Indicate the equivalence relationship.

6. Let $X$ be a set and $R$ be an equivalence relation on $X$. Define a function $k: X \to \mathcal{P}(X)$ by $k(x) = \{y \in X : xRy\}$. Prove that $k$ is bijective.

Solution

- Since $h$ is injective and $g$ is surjective, $k = h \circ g$ is both injective and surjective, so $k$ is bijective.

Diagram:

- Draw a Venn diagram showing $X$.
- Label $x$ and $Y$.
- Indicate the containment relationship.

7. Let $X$ be a set and $R$ be an equivalence relation on $X$. Define a function $l: X \to \mathcal{P}(X)$ by $l(x) = \{y \in X : xRy\}$. Prove that $l$ is a bijection.

Solution

- Since $h$ is injective and $g$ is surjective, $l = h \circ g$ is both injective and surjective, so $l$ is a bijection.

Diagram:

- Draw a Venn diagram showing $X$.
- Label $x$ and $Y$.
- Indicate the containment relationship.