Review of Probability

Applied Machine Learning

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What is a probability

- A belief, a confidence, a likelihood

- “There’s a 60% chance it will rain tomorrow.”
  - Based on the information I have, if we were to simulate the future 100 times, I’d expect it to rain 60 of them.
  - I think it’s a little more likely to rain than not

- You have a 1/18 chance of rolling a 3 with two dice.
  - If you roll an infinite number of pairs of dice, 1 out of 18 of them will sum to 3.

- Probabilities are expectations, according to some information and assumptions.
  - E.g., it will either rain tomorrow or not

Why do we care about probability in machine learning?

- ML problems are often formulated as maximizing a conditional probability $P(y|\boldsymbol{X})$, e.g. the probability of the true label given features, or maximizing the data likelihood $P(\boldsymbol{X})$

- Algorithms involving probabilistic objectives include logistic regression, naïve Bayes, decision trees, boosting, random forests, deep networks, EM algorithm, and more

Example

There are two movies showing: "Bumblebee", with 40 attendees, and "Apocalypse" with 60 attendees.

What is the probability that a random person is watching Apocalypse?
P(X=A)

- Out of all events, what fraction satisfy the criterion

Movie	Attendees
Bumblebee	40
Apocalypse	60

100 total people

Of those 60 are watching Apocalypse

So $P(M = A) = 60 \ / \ 100 \ = \ 0.6$

Joint Probability

Suppose we also know whether each movie-goer is a child or adult

What is the probability that a movie-goer watches Bumblebee and is an adult?

- Out of all events, what fraction satisfy all criteria

Movie	Adult	Child
Bumble bee	20	20
Apocal ypse	50	10

$$P(M = B, Age = Adult) = 20 / (20 + 20 + 50 + 10) = 0.2$$

Conditional Probability

Movie	Adult	Child
Bumble bee	20	20
Apocal ypse	50	10

Given that a movie-goer is a child, what is the probability that he or she is watching Bumblebee?

– Out of all events that satisfy the condition, what fraction satisfy the criterion

$$P(M = B \mid Age = Child) = 20 / (20 + 10) = 0.667$$

Conditional Probability

If I know a movie-goer watching Bumblebee, what is the probability he or she is a child?

- Out of all events that satisfy the condition, what fraction satisfy the criterion

Movie	Adult	Child
Bumblebee	20	20
Apocalypse	50	10

$$P(M = B \mid Age = Child) = 20 \,/\, (20 + 20) = 0.5$$

Relationships between joint and conditional/marginal probabilities

Joint probability is the product of the conditional probability and the probability that the condition is true.

$$P(X,Y) = P(X|Y)P(Y) = P(Y|X)P(X)$$

This extends to many variables with a chain rule.

$$P(X,Y,Z) = P(X|Y,Z)P(Y|Z)P(Z)$$

Marginalize out a variable by summing over its possible values

$$P(X) = \sum_i P(X, Y = y_i) = \sum_i P(X|Y = y_i)P(y = y_i)$$

$$P(Movie, Age)$$

	Adult	Child
Bumblebee	0.2	0.2
Apocalypse	0.5	0.1

From the joint probability table, you can always compute probabilities of subsets of variables or conditional probabilities with those variables.

$P(Adult) = 0.2 + 0.5$ $P(Bumblebee) = 0.2 + 0.2$ $P(Adult \mid Bumblebee) = 0.2/(0.2 + 0.2)$

A is independent of B if (and only if)

$$P(A, B) = P(A)P(B)$$

$$P(A|B) = P(A), \qquad P(B|A) = P(B)$$

A and B are conditionally independent of C if (and only if)

$$P(A, B|C) = P(A|C)P(B|C)$$

Estimate a discrete probability function by counting

Movie	Age
B	C
B	A
A	A
A	A
B	A
A	A
…	

$$P(M = B) = \frac{1}{|N|}\sum_{n} \delta(M_n = B)$$

$$P(M = B, A = C) = \frac{1}{|N|}\sum_{n} \delta(M_n = B, A_n = C)$$

Probability density functions of continuous variables

What if the variable is continuous?

Precise values may never occur and be infinitesimally unlikely, e.g.

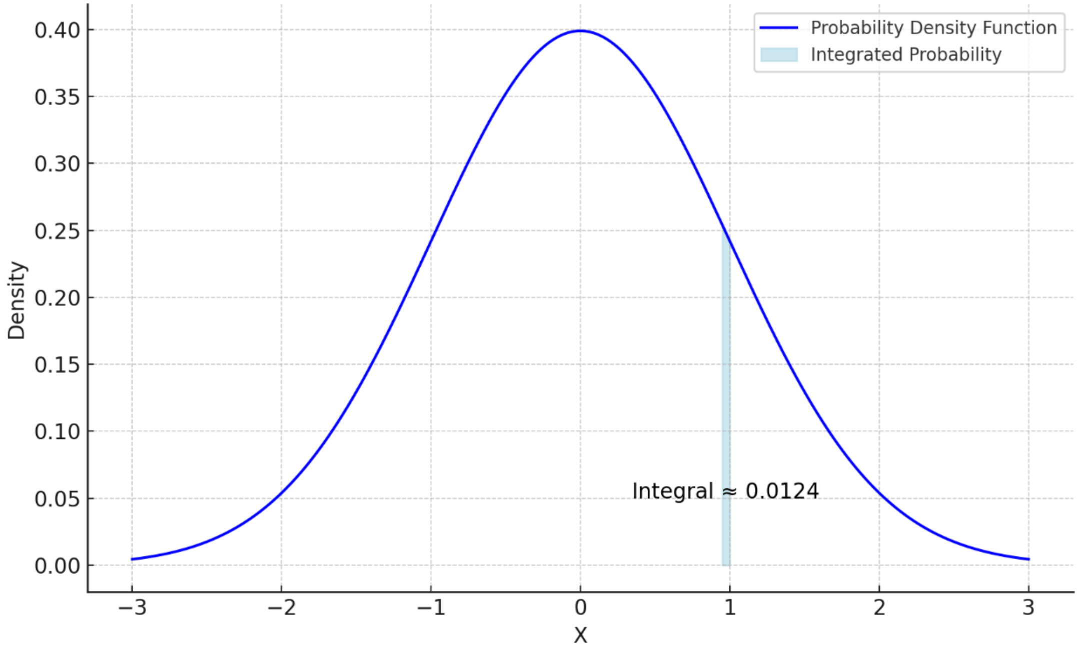
$$P(Age = 11.05) = 0$$

We replace the probability function with the probability *density* function, which can be integrated over a range to give a probability.

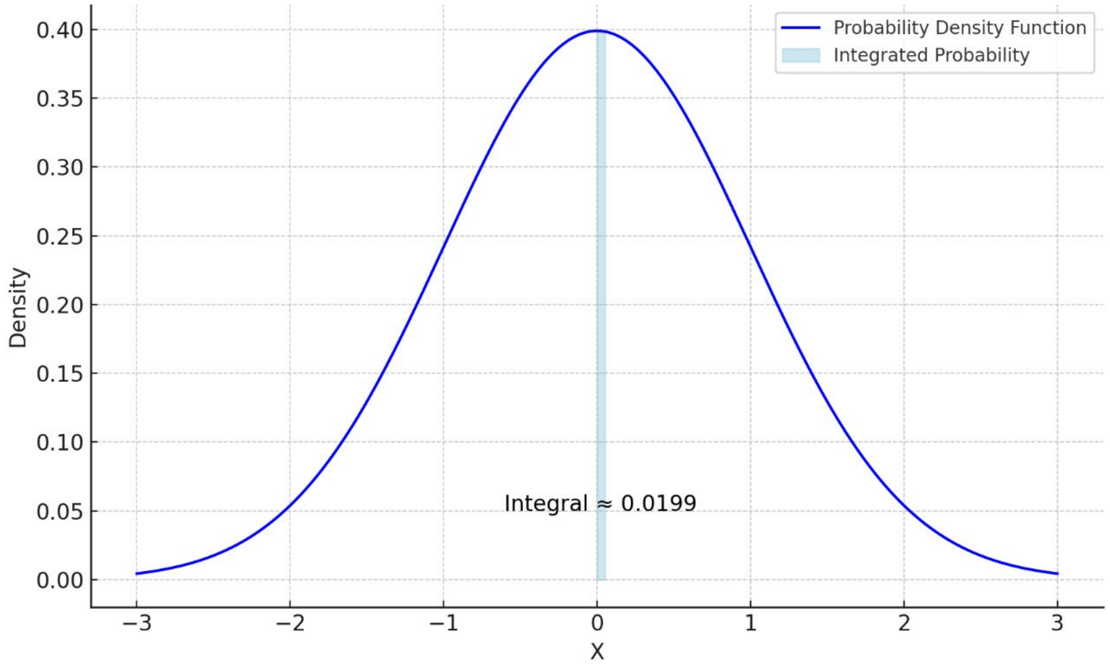
$$P(10 \leq Age < 11) = \int_{10}^{11} p(Age = a)da$$

Movie	Age
B	10
B	38
A	22
A	19
B	25
A	50
…	

Probability density function (PDF) → Probability



$P(0.95 < X < 1) \approx 0.012$



$P(0 < X < 0.05) \approx 0.02$

$P(-0.05 < X < 0.05) \approx 0.04$

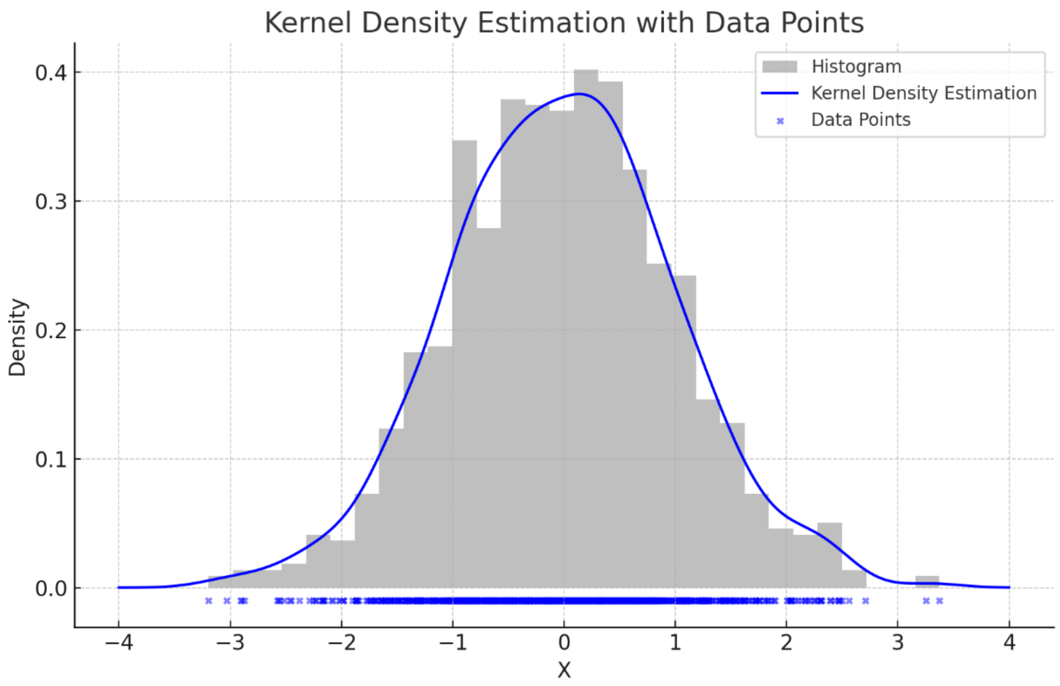
How to estimate probability density functions from samples

1. Discretize and count (histogram)

2. Kernel density estimation

3. Fit parameters of a model

Gaussian: $\mu = -0.015 \;\; \sigma = 1.004$



What must be true about probability functions?

1. Probabilities cannot be negative

2. The sum (for discrete) or integral (for continuous) must be 1
    - For discrete, this means that each value must be between 0 and 1
    - But values can be greater than 1 in a probability density function

Expectation and Variance

- The *expected value* or *mean* of a random variable $X \in \{v_{1,} \ldots, v_N\}$ is the average value we'd get if we took an infinitely large sample:

$$E(X) = \lim_{M\to\infty} \frac{1}{M} \sum_{x_i \sim P(X)} x_i = \sum_{i=1..N} P(X = v_i) v_i$$

- We can take the expectation of a function $f(X)$:

$$E\big(f(X)\big) = \sum_{i=1..N} P(X = v_i) f(v_i)$$

- The *variance* of X measures the expected square difference of the values from the mean

$$\mathrm{Var}(X) = E\big((E(X) - X)^2\big) = \sum_{i=1..N} P(X = v_i)(E(x) - v_i)^2$$

- We can also take the mean or variance of a sample $\mathrm{S} = [s_0, \ldots, s_K]$, called the empirical mean or variance

$$E(S) = \frac{1}{K} \sum_{i=1..K} s_i \qquad \mathrm{Var}(S) = \frac{1}{K} \sum_{i=1..K} (E(S) - s_i)^2$$

How do we measure the amount of data required to store a value of a variable?

- First, let's consider the likelihood of a set of values $\boldsymbol{x} \sim P(X)$: $\mathrm{P}(\boldsymbol{x}) = \prod_i P(x_i)$, assuming the values are independent. However, this will become inconveniently small, so we can equivalently consider $\log P(\boldsymbol{x}) = \sum_i \log P(x_i)$. The expected value of this log likelihood gives us a measure of predictability.

- Entropy $H(X)$ is the expected negative log likelihood of variable $X \in \{v_1, \dots, v_N\}$

$$H(X) = \sum_{i=1..N} -P(X = v_i) \log_2 P(X = v_i)$$

- Greater entropy means that the value of X is less predictable
  - If $H(X) = 0$, then $P(X = x_i) = 1$ for some x_i (and 0 for others)
  - If $H(X) = \log_2 P(N)$, then $P(X = x_i) = \frac{1}{N}$ for all N values

- If the values are less predictable, more bits are needed. Shannon's source coding theorem shows that the minimum expected number of bits required to encode a string of K i.i.d. values sampled from $P(X)$ is $KH(X)$
  - Complicated to prove, but think of each bit as dividing the possible values into two equally likely sets
  - E.g. let P(X=1)=1/4, P(X=2)=1/4, P(X=3)=0, P(X=4)=1/2. One bit can split ([1,2], 4) and, if needed, a second bit can split between 1 and 2. To encode, 1→00, 2→01, 4→1. This requires 1.5 bits on average. $H(X) = 0.25 * 2 + 0.25 * 2 + 0.5 * 1 = 1.5$. 1-1-1-01-00-10-1 = 4,4,4,2,1,4,2

$x \sim P(X)$ means that x is drawn (randomly sampled) from the distribution $P(X)$

Typical machine learning problem: predict Y from X

- Given some features X, we want to predict target variable y, e.g.
  - X = email text and header; y= spam or not spam
  - X = meteorological data; y = next day's high temperature
  - X = image of a handwritten number; y = 0, 1, …, or 9

- We often frame this probabilistically
  - To predict, select $\mathrm{y}^* = \mathrm{argmax}_y\, P(y|X)$, i.e. choose the y that is most likely given X
  - To train, optimize parameters that maximize the likelihood of the labels of the training data given the features of the training data

Basics of vector/matrix multiplication

- $\boldsymbol{w}^T \boldsymbol{x} = \boldsymbol{w} \cdot \boldsymbol{x} = \sum_i w_i x_i$

- Element (i, j) of AB is the dot product of the ith row of A with the jth column of B

- $AB \neq BA$

- If A is $N \times M$ size matrix and B is $M \times K$, then AB is $N \times K$

- If A is $N \times M$ size matrix and B is $L \times K$ with $L \neq M$ then A cannot be multiplied by B

Partial derivatives

$$\frac{\partial}{\partial w_i}\boldsymbol{w}^T\boldsymbol{x} = \frac{\partial}{\partial w_i}\sum_i w_i x_i = x_i$$

Classification by maximizing label likelihood

Suppose we want to predict a label $y_i \in \{-1, 1\}$ from an image X_i. Given a set of N training examples, solve for model parameters $\boldsymbol{w}$ to maximize $P(y_1 \dots y_n | X_1 .. X_N)$. I.e., find the model parameters that make the training labels most likely, given the training features.

1. Assume each training sample is a sample from an identical and independent distribution (iid assumption): $P(y_1 \dots y_n | X_1 .. X_N) = \prod_{i=1..N} P(y_i | X_i)$

2. Maximizing a product is hard because the derivative is complicated. Instead, we can maximize a sum of logs

$$\log \prod_{i=1..N} P(y_i | X_i) = \sum_{i=1..N} \log P(y_i | X_i)$$

3. We need a function (a.k.a. a model) to output the probability given the label. Let's use linear logistic regression

$$f(X_i, \boldsymbol{w}) = \boldsymbol{w}^T X_i = \log P(y_i = 1 | X_i) - \log P(y\_i = -1 | X_i) = \log \frac{P(y_i = 1 | X_i)}{P(y_i = -1 | X_i)}$$

4. This is called a logistic score or logit. We can convert the logit to a probability:

$$\frac{1}{1 + \exp\left(-\log \frac{P(y_i = 1 | X_i)}{P(y_i = -1 | X_i)}\right)} = \frac{1}{1 + \frac{P(y_i = -1 | X_i)}{P(y_i = 1 | X_i)}} = \frac{P(y_i = 1 | X_i)}{P(y_i = 1 | X_i) + P(y_i = -1 | X_i)} = P(y_i = 1 | X_i)$$

5. This function $\sigma(x) = 1/(1 + \exp(-x))$ is called a sigmoid. So $\sigma(\boldsymbol{w}^T X_i) = P(y_i = 1 | X)$. Also, $\sigma(-\boldsymbol{w}^T X_i) = P(y_i = -1 | X)$.

6. Now we can write our objective in terms of parameters, image features, and labels:

$$\boldsymbol{w}_{opt} = \operatorname{argmax}_{\boldsymbol{w}} \sum_i \log(y_i \sigma(\boldsymbol{w}^T X_i)) = \operatorname{argmin}_{\boldsymbol{w}} \sum_i -\log(y_i \sigma(\boldsymbol{w}^T X_i))$$

7. The argmin expression is the "loss". We can optimize by taking the derivative of this expression wrt $\boldsymbol{w}$ and performing gradient descent.

Problems

https://us.prairielearn.com/pl/course_instance/157430/assessment/2432153