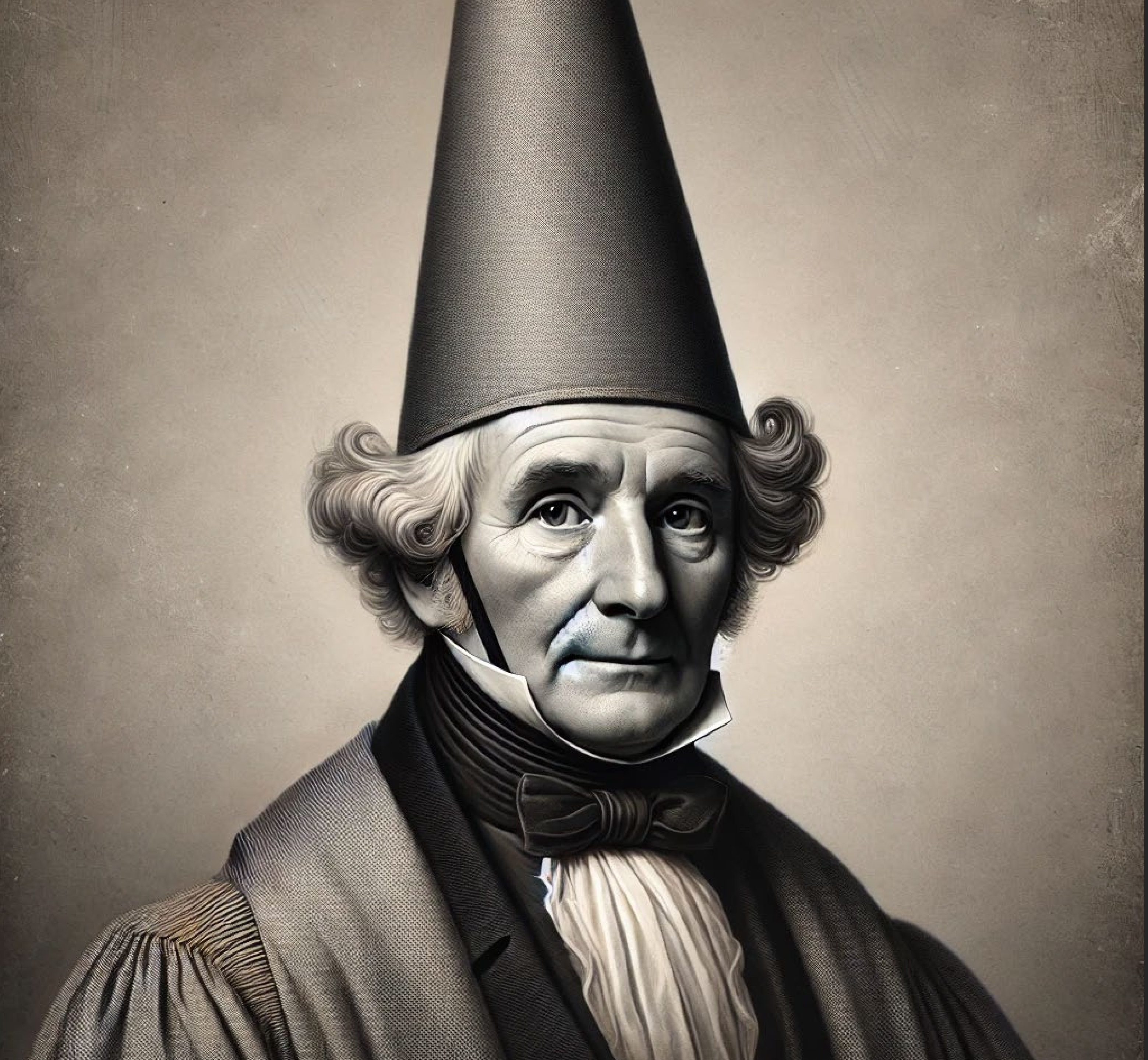


Naïve Bayes Classifier

Applied Machine Learning
Derek Hoiem

Dall-E: portrait of Thomas Bayes with a Dunce Cap on his head

Recap of approaches we’ve seen so far

- Nearest neighbor is widely used
  - Super-powers: can instantly learn new classes and predict from one or many examples
- Logistic Regression is widely used
  - Super-powers: Effective prediction from high-dimensional features
- Linear Regression is widely used
  - Super-powers: Can extrapolate, explain relationships, and predict continuous values from many variables
- Almost all algorithms involve nearest neighbor, logistic regression, or linear regression
  - The main learning challenge is typically **feature learning**

Today’s Lecture

- Introduce probabilistic models
- Naïve Bayes Classifier
  - Assumptions / model
  - How to estimate from data
  - How to predict given new features
- “Semi-naïve Bayes” object detector

What is a probability

- A belief, a confidence, a likelihood

- “There’s a 60% chance it will rain tomorrow.”
  - Based on the information I have, if we were to simulate the future 100 times, I’d expect it to rain 60 of them.
  - I think it’s a little more likely to rain than not

- You have a 1/18 chance of rolling a 3 with two dice.
  - If you roll an infinite number of pairs of dice, 1 out of 18 of them will sum to 3.

- Probabilities are expectations, according to some information and assumptions.
  - E.g., it will either rain tomorrow or not

Joint and conditional probability

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

$$P(a, b, c) = P(a|b, c)P(b|c)P(c)$$

Bayes Rule: $$P(x|y) = \frac{P(x, y)}{P(y)} = \frac{P(y|x)P(x)}{P(y)}$$

Law of total probability $$\left[\sum_{v \in x} P(x = v)\right] = 1$$

Marginalization $$\left[\sum_{v \in x} P(x = v, y)\right] = P(y)$$

For continuous variables, replace sum over possible values with integral over domain

Estimate probabilities of discrete variables by counting

$$P(x = v) = \frac{1}{|N|} \sum_{n} \delta(x_n = v)$$

Example

x: Larger than 10 lbs?

y		F	T
	Cat	15	25
	Dog	5	40

$$P(y = Cat) =$$

$$P(y = Cat|x = F) =$$

$$P(x = F|y = Cat) =$$

Example

x: Larger than 10 lbs?

y		F	T
	Cat	15	25
	Dog	5	40

$$P(y = Cat) = 40\ /\ 85$$

$$P(y = Cat | x = F) = 15/20$$

$$P(x = F | y = Cat) = 15/40$$

A is independent of B if (and only if)

$$P(A, B) = P(A)P(B)$$

$$P(A|B) = P(A), \qquad P(B|A) = P(B)$$

What if you have 100 variables? How can you count all combinations?

Fully modeling dependencies between many variables (more than 3 or 4) is challenging and requires a lot of data

Probabilistic model

$$y^* = \underset{y}{\operatorname{argmax}}\, P(y|x)$$

Or equivalently…

$$y^* = \underset{y}{\operatorname{argmax}}\, P(x|y)P(y)$$

$$\underset{y}{\operatorname{argmax}}\, P(y|x) = \underset{y}{\operatorname{argmax}}\, P(y|x)P(x) = \underset{y}{\operatorname{argmax}}\, P(y,x) = \underset{y}{\operatorname{argmax}} P(x|y)P(y)$$

Notation

- x_i is the ith feature variable
  - i indicates the feature index
- x_n is the nth feature vector
  - n indicates the sample index
- y_n is the nth label
- x_{ni} is the ith feature of the nth sample
- $\delta(x_{ni} = v)$ returns 1 if $x_{ni} = v$; 0 otherwise
  - v indicates a feature value
  - δ is an indicator function, mapping from true/false to 1/0

Naïve Bayes Classifier

https://www.kaggle.com/datasets/uciml/sms-spam-collection-dataset

Suppose you want to classify whether a text message is spam

ham
Go until jurong point, crazy.. Available only in bugis n great world la e buffet... Cine there got a...

ham
Ok lar... Joking wif u oni...

spam
Free entry in 2 a wkly comp to win FA Cup final tkts 21st May 2005. Text FA to 87121 to receive entr...

ham
U dun say so early hor... U c already then say...

ham
Nah I don't think he goes to usf, he lives around here though

spam
FreeMsg Hey there darling it's been 3 week's now and no word back! I'd like some fun you up for it s...

Naïve Bayes Classifier

https://www.kaggle.com/datasets/uciml/sms-spam-collection-dataset

Suppose you want to classify whether a text message is spam or not

P("Ok lar... Joking wif u oni...") probably is 0 in the training set because you might not get exactly the same message twice

How to model?

Naïve Bayes Model

Assume features $x_1 .. x_m$ are independent given the label y:

$$P(\boldsymbol{x}|y) = \prod_i P(x_i|y)$$

Then

$$y^* = \underset{y}{\operatorname{argmax}} \prod_i P(x_i|y) P(y)$$

$$= \underset{y}{\operatorname{argmax}} [\log P(y) + \sum_i \log P(x_i|y)]$$

Naïve Bayes Classifier

https://www.kaggle.com/datasets/uciml/sms-spam-collection-dataset

1. Estimate $P(y = spam)$ and $P(y = ham)$
2. Estimate $P(word = j \mid spam)$ and $P(word = j \mid ham)$

$$P(\text{“}Ok\ lar\ \ldots\ Joking\ wif\ u\ oni\ \ldots\text{”}, spam)$$
$$=\ P(spam)P(\text{“}Ok\text{”} \mid spam)P(\text{“}lar\text{”}|spam)\ \ldots P(\text{“}oni\text{”}|spam)$$

$$P(\text{“}Ok\ lar\ \ldots\ Joking\ wif\ u\ oni\ \ldots\text{”}, ham)$$
$$=\ P(ham)P(\text{“}Ok\text{”} \mid ham)P(\text{“}lar\text{”}|ham)\ \ldots P(\text{“}oni\text{”}|ham)$$

If $P(message, spam)\ >\ P(message, ham)$, it’s more likely spam.

Naïve Bayes Algorithm

- Training
  1. Estimate parameters for $P(x_i|y)$ for each i
  2. Estimate parameters for $P(y)$
- Prediction
  1. Solve for y that maximizes $P(x, y)$

$$y^* = \underset{y}{\operatorname{argmax}} \prod_i P(x_i|y)P(y)$$

Naïve Bayes Algorithm

- Training
  1. Estimate parameters for $P(x_i|y)$ for each i
  2. Estimate parameters for $P(y)$
- Prediction
  1. Solve for y that maximizes $P(x, y)$

$$y^* = \underset{y}{\operatorname{argmax}} \prod_i P(x_i|y)P(y)$$

But generally, $P(x, y)$ will get vanishingly small if x contains many features, so instead compute $\log P(x, y)$

$$y^* = \underset{y}{\operatorname{argmax}}[\, \log P(y) + \sum_i \log P(x_i|y)\,]$$

For spam detection, the spam score is
$\log P(x, y = spam) - \log P(x, y = ham)$

How to estimate $P(x_i|y)$ from data?

1. MLE (maximum likelihood estimation): Choose the parameter that maximizes the likelihood of the data

2. MAP (maximum a priori): Choose the parameter that maximizes the data likelihood and its own prior

MLE (maximum likelihood estimation)

- MLE: Choose the parameter that maximizes the likelihood of the data
- Bernoulli (x is binary; y is discrete)

$$P(x_i|y = k) = \theta_{ki}^{x_i}(1 - \theta_{ki})^{1-x_i}$$

$$\theta_{ki} = \sum_n \delta(x_{ni} = 1, y_n = k) / \sum_n \delta(y_n = k)$$

```
theta_ki[k,i] = np.sum((X[:,i]==1) & (y==k)) / np.sum(y==k)
```

- Categorical (x is has multiple discrete values, y is discrete)

$$\theta_{kiv} = \sum_n \delta(x_{ni} = v, y_n = k) / \sum_n \delta(y_n = k)$$

```
theta_kiv[k,i,v] = np.sum((X[:,i]==v) & (y==k)) / (np.sum(y==k))
```

Priors and MAP (maximum a priori)

- MAP: Choose the parameter that maximizes the data likelihood and its own prior
- Priors on the likelihood parameters prevent a single feature from having zero or extremely low likelihood due to insufficient training data
    - As Warren Buffet says, it's not just about maximizing expected return – it's about making sure there are no zeros.

- Discrete: initialize counts with α (e.g. $\alpha = 1$)

$$P(x_i = v|y = k) = \frac{\alpha + count(x_i = v, y = k)}{\sum_v[\alpha + count(x_i = v, y = k)]}$$

```
theta_kiv[k,i,v] = (np.sum((X[:,i]==v) & (y==k))+alpha) / (np.sum(y==k)+alpha*num_v)
```

MLE and MAP estimates of binary variable likelihoods

- MLE (maximize data likelihood)

$$P(x = 1|y = 1) = \frac{\sum_n \delta(x_n = 1, y_n = 1)}{\sum_n \delta(x_n = 0, y_n = 1) + \sum_n \delta(x_n = 1, y_n = 1)}$$

- MAP (maximum a posteriori) with prior α

$$P(x = 1|y = 1) = \frac{\alpha + \sum_n \delta(x_n = 1, y_n = 1)}{(\alpha + \sum_n \delta(x_n = 0, y_n = 1)) + (\alpha + \sum_n \delta(x_n = 1, y_n = 1))}$$

- This is a Bayesian prior that implies $P(x = 0|y) \approx P(x = 1|y)$, unless data tells us differently
- Similar concept to regularization that we saw in linear regression and classification
- Important because it avoids zeros that could dominate the overall likelihood and provides a more stable estimate with limited data
- With more data, the prior has less effect

Q1-Q3

https://tinyurl.com/441-fa24-L8

What if $\boldsymbol{x}$ is continuous? Can we still use Naïve Bayes?

- E.g., estimate whether a person is male or female based on height and weight

$$P(m = 1|h, w) = \frac{P(m = 1, h, w)}{P(h, w)}$$

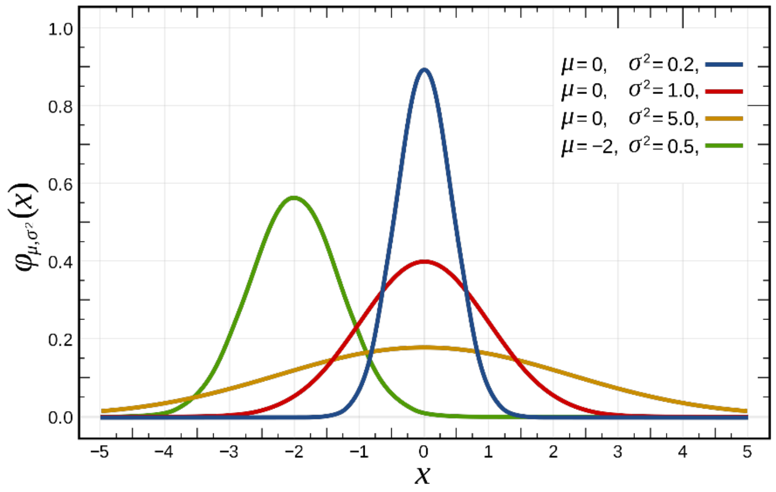
$$P(m = 1, h, w) = P(m = 1)P(h|m = 1)P(w|m = 1)$$

$$P(m = 0, h, w) = P(m = 0)P(h|m = 0)P(w|m = 0)$$

$$P(h, w) = P(m = 1, h, w) + P(m = 0, h, w)$$

Suppose x_i is Gaussian, and y is discrete

$$P(x_i \mid y=k) = \frac{1}{\sqrt{2\pi}\,\sigma_{ki}} \cdot \exp\left(-\frac{1}{2}\frac{(x_i - \mu_{ki})^2}{\sigma_{ki}^2}\right)$$



$$\mu_{ki} = \sum_n \left[x_{ni} \cdot \delta(y_n = k)\right] / \sum_n \delta(y_n = k)$$

$$\sigma_{ki}^2 = \sum_n \left[(x_{ni} - \mu_{ki})^2 \cdot \delta(y_n = k)\right] / \sum_n \delta(y_n = k)$$

```
mu[k,i] = np.mean(X[y==k:,i], axis=0)
sigma[k, i] = np.std(X[y==k,i], axis=0)
```

Prior for Gaussian distributions

- Add some ϵ to the variance (e.g. $\epsilon = 0.1/N$)
  - For multivariate, add to diagonal of covariance

```
sigma[k, i] = np.std(X[y==k,i], axis=0) + np.sqrt(0.1/len(X))
```

Suppose we want to predict weight from height and m/f?

- $P(w, h, m) \; = \; P(w|h)P(m|w)P(w)$

- Less convenient, but
  - $P(w|h) = P(h, w) \; / \; P(w)$, which can be modeled with a 2D Gaussian and a 1D Gaussian
  - $P(m = 1|h) = P(h|m = 1)P(m = 1)/P(h)$, which can each be modeled with 1D Gaussian or categorical
  - $P(m = 1|h) = \frac{P(h|m = 1)P(m=1)}{P(h)}$
  - $w^{*} = \mathrm{argmax_w} P(w, h, m)$ can be solved by $0 = \frac{\partial}{\partial w} P(w, h, m)$, which will have a closed form solution in this case

How to predict y from x when $(y - x_i)$ is Gaussian

If y is continuous,

$$\frac{\partial}{\partial y} \sum_i \log P(x_i|y) + \log P(y) = 0$$

General formulation (set partial derivative wrt y of $\log P(x, y)$ to 0)

$$\frac{\partial}{\partial y} \sum_i -\frac{1}{2}\frac{(y-x_i-\mu_i)^2}{\sigma_i^2} - \frac{1}{2}\frac{(y-\mu_y)^2}{\sigma_y^2} = 0$$

Example of Temperature regression: $y - x_i$ is Gaussian

$$\frac{\partial}{\partial y} \sum_i \frac{1}{2}\frac{-y^2}{\sigma_i^2} + \frac{yx_i}{\sigma_i^2} + \frac{y\mu_i}{\sigma_i^2} \quad \frac{-1y^2}{2\sigma_y^2} + \frac{y\mu_y}{\sigma_y^2} = 0$$

$$\sum_i \left(\frac{-y}{\sigma_i^2} + \frac{x}{\sigma_i^2} + \frac{\mu_i}{\sigma_i^2}\right) - (y-\mu_y)/\sigma_y^2 = 0$$

$$y\left(\sum_i \frac{1}{\sigma_i^2} + \frac{1}{\sigma_y^2}\right) = \sum_i \frac{x+\mu_i}{\sigma_i^2} + \frac{\mu_y}{\sigma_y^2}$$

$$y = \frac{1}{\sum_i \frac{1}{\sigma_i^2} + \frac{1}{\sigma_y^2}} \cdot \left[\sum_i \frac{x+\mu_i}{\sigma_i^2} + \frac{\mu_y}{\sigma_y^2}\right]$$

$$y = \frac{1}{\sum_i w_i + w_y}\left[\sum_i (x+\mu_i)w_i + \mu_y \cdot w_y\right]$$

Prediction is weighted average of means, where weights are inverse variance

$$P(x_i|y) \sim N\left(y - x_i, \sigma^2\right) = \frac{1}{\sqrt{2\pi}\,\sigma_i} \exp\left(-\frac{1}{2}\frac{(y-x_i-\mu_i)^2}{\sigma_i^2}\right)$$

Q4-Q6

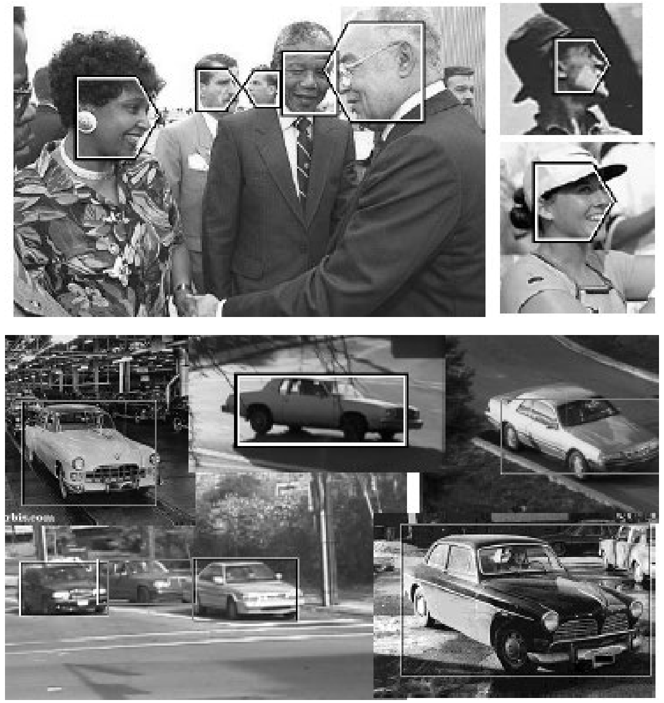
https://tinyurl.com/441-fa24-L8

Use case: “Semi-naïve Bayes” object detection

- Best performing face/car detector in 2000-2005
- Model probabilities of small groups of features (wavelet coefficients)
- Search for groupings, discretize features, estimate parameters

A Statistical Method for 3D Object Detection Applied to Faces and Cars

Henry Schneiderman and Takeo Kanade



$$\frac{\prod_{x,y\in \text{region}} \prod_{k=1}^{17} P_k(pattern_k(x,y),x,y\,|\,\text{object})}{\prod_{x,y\in \text{region}} \prod_{k=1}^{17} P_k(pattern_k(x,y),x,y\,|\,\text{non-object})} > \lambda$$

https://www.cs.cmu.edu/afs/cs.cmu.edu/user/hws/www/CVPR00.pdf

Naïve Bayes

- Pros
  - Easy and fast to train
  - Fast inference
  - Can be used with continuous, discrete, or mixed features
- Cons
  - Does not account for feature interactions
  - Does not provide good confidence estimate
- Notes
  - Best when used with discrete variables, variables that are well fit by Gaussian, or kernel density estimation

Things to remember

- Probabilistic models are a large class of machine learning methods

- Naïve Bayes assumes that features are independent given the label
  - Easy/fast to estimate parameters
  - Less risk of overfitting when data is limited

$$P(\boldsymbol{x}, y) = \prod_i P(x_i|y)P(y)$$

- You can look up how to estimate parameters for most common probability models
  - Or take partial derivative of total data/label likelihood given parameter

- Prediction involves finding y that maximizes $P(x, y)$, either by trying all y or solving partial derivative

- Maximizing $\log P(x, y)$ is equivalent to maximizing $P(x, y)$ and often much easier

$$y^* = \underset{y}{\operatorname{argmax}} \prod_i P(x_i|y)\, P(y)$$

$$= \underset{y}{\operatorname{argmax}} \sum_i \log P(x_i|y) + \log P(y)$$

Next week

- EM and Density Estimation