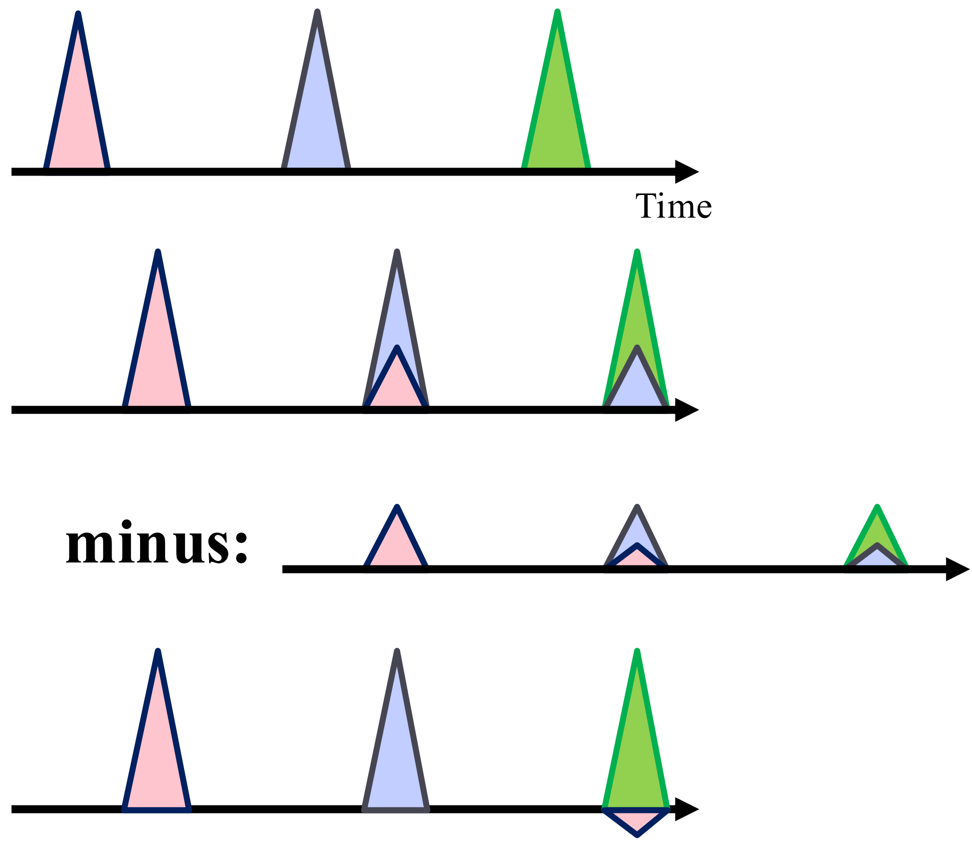
CS/ECE 439: Wireless Networking

Physical Layer - Diversity

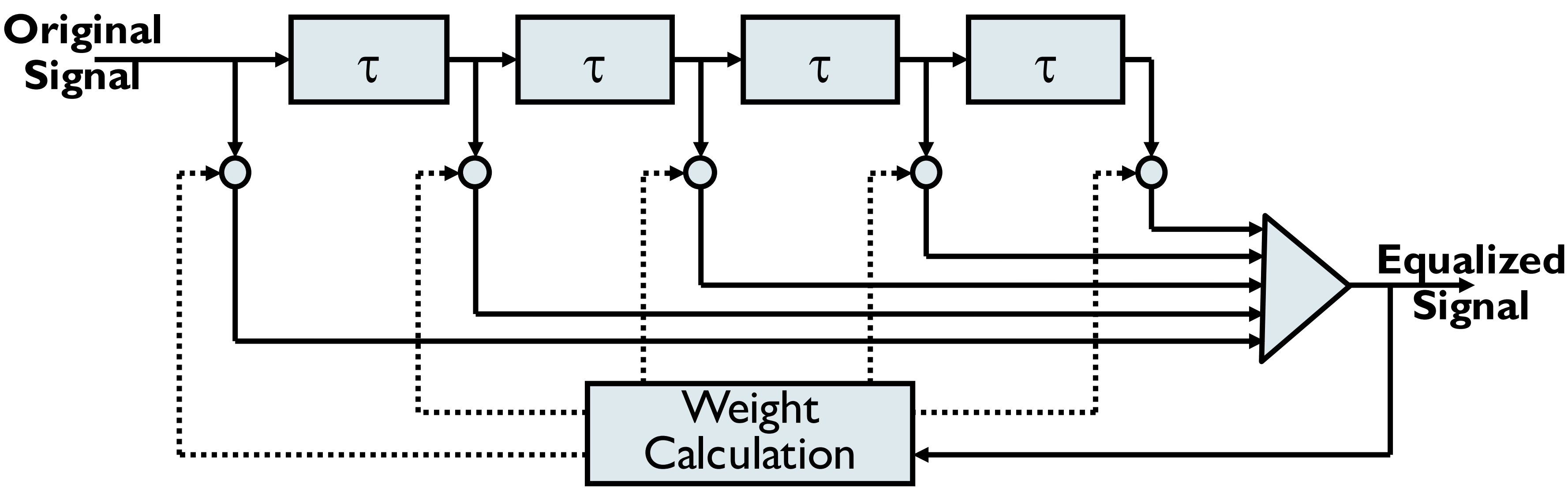
Inter-Symbol Interference

- Larger difference in path length can cause inter-symbol interference (ISI)
- Suppose the receiver can do some processing
  - Add/subtracted scaled and delayed copies of the signal

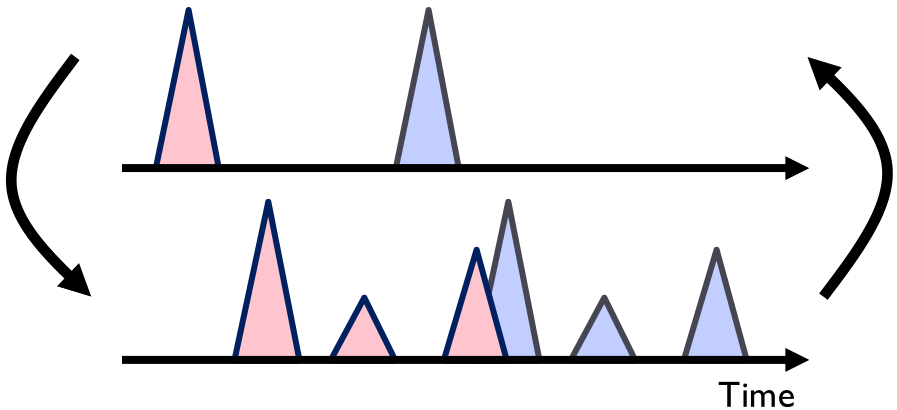


Dynamic Equalization

- Combine multiple delayed copies of the signal
  - ex: linear equalizer circuit



Equalization Discussion



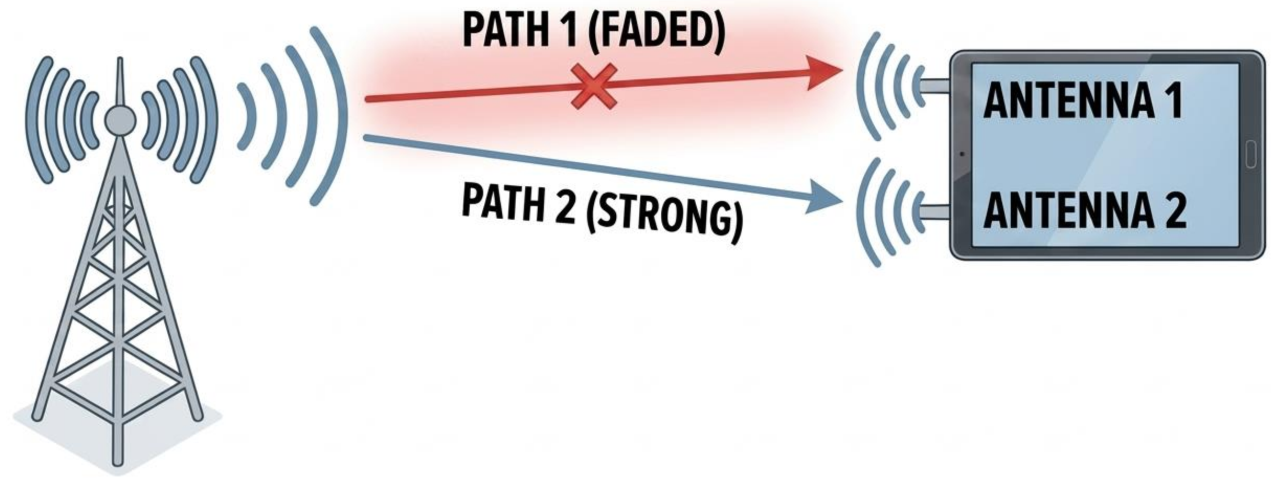
- Use multiple delayed copies of the received signal to try to reconstruct the original signal
- Weights are set dynamically
  - Typically based on some known "training" sequence
- Effectively uses the multiple copies of the signal to reinforce each other
  - But only works for paths that differ in length by less than the depth of the pipeline

Diversity Techniques

- Spatial diversity
- Frequency diversity
- Time diversity

Spatial Diversity

- Use multiple antennas that pick up the signal in slightly different locations
  - Can use more than two antennas!

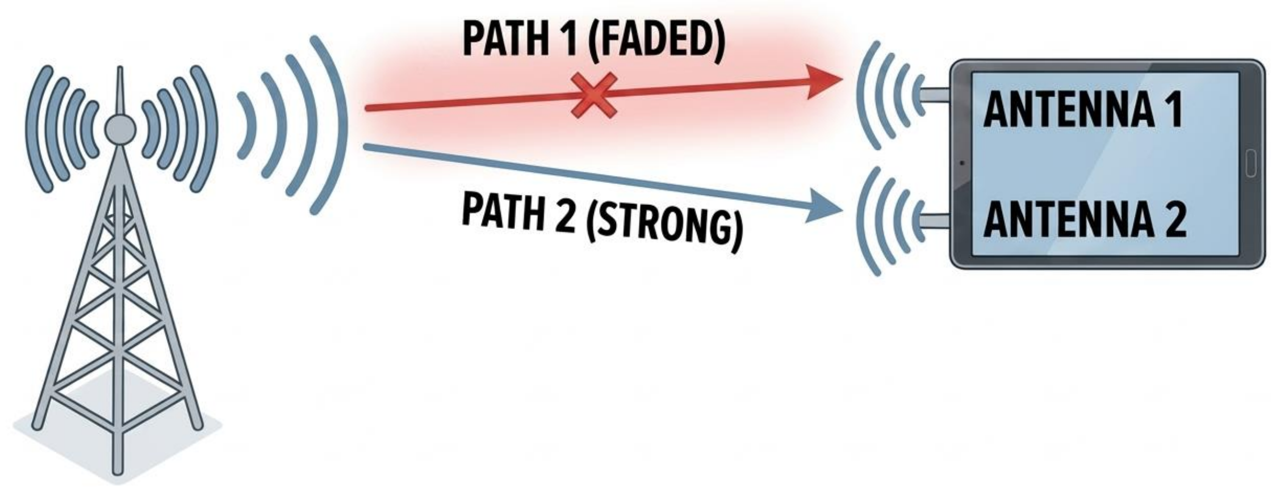


Spatial Diversity

- Use multiple antennas that pick up the signal in slightly different locations
  - Can use more than two antennas!
- Each antenna experiences different channels
  - If antennas are sufficiently separated, chances are that the signals are mostly uncorrelated
  - If one antenna experiences deep fading, chances are that the other antenna has a strong signal
    - Antennas should be separated by ½ wavelength or more
- Applies to both transmit and receive side
  - Channels are symmetric

Receiver Diversity

- Simplest solution
  - Selection diversity: pick antenna with best SNR



Receiver Diversity

- Simplest solution
  - Selection diversity: pick antenna with best SNR

- But why not use both signals?

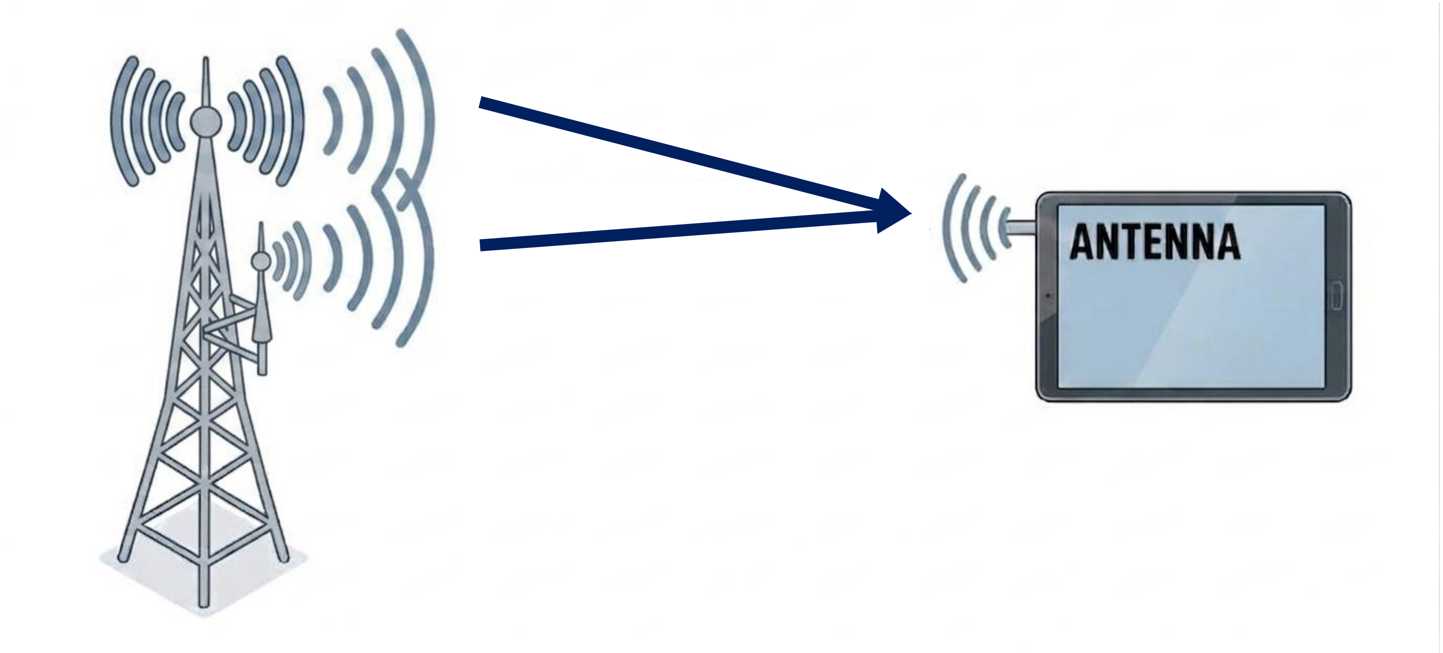
  \+ More information

  \- Signals out of phase, e.g. kind of like multi-path

  ? Don't amplify the noise
- Maximal ratio combining: combine signals with a weight that is based on their SNR
  - Weight will favor the strongest signal (highest SNR)

Transmit Diversity

- Same as receive diversity but the transmitter has multiple antennas

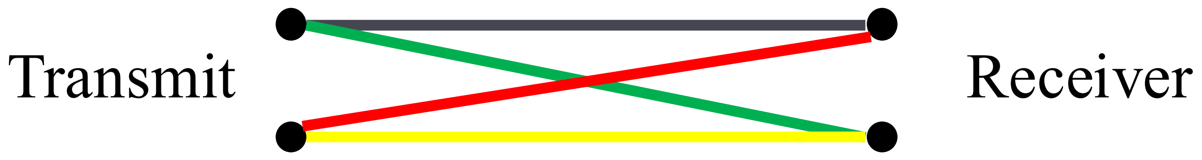


Transmit Diversity

- Same as receive diversity but the transmitter has multiple antennas
- Selection diversity: transmitter picks the best antenna
  - i.e. with best channel to receiver
  - Sender "precodes" the signal
- How does transmitter learn channel?
  - Gets explicit feedback from the receiver
  - Rely on channel reciprocity

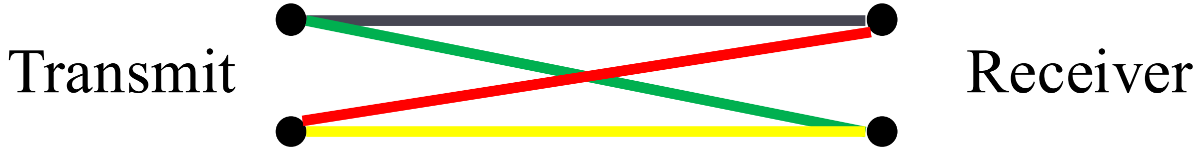
Typical Algorithm in 802.11

- Use transmit + receive selection diversity
- How to explore all channels to find the best one … or at least the best transmit antenna
- Receiver
  - Use the antenna with the strongest signal
  - Always use the same antenna to send the acknowledgement – gives feedback to the sender



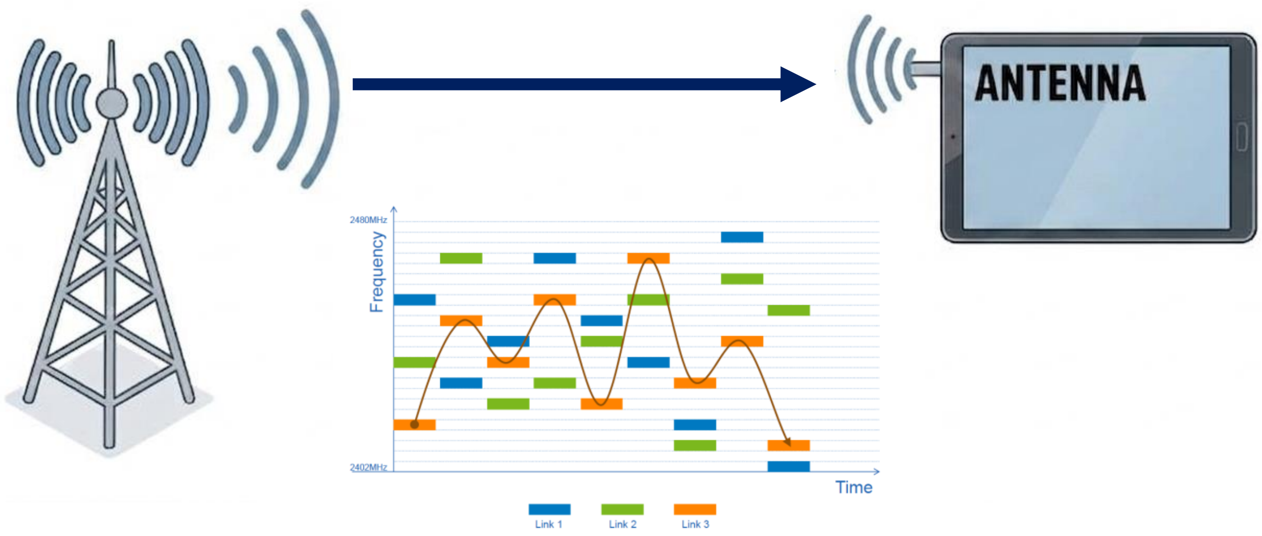
Typical Algorithm in 802.11

- Use transmit + receive selection diversity
- How to explore all channels to find the best one … or at least the best transmit antenna
- Sender
  - Pick an antenna to transmit and learn about the channel quality based on the ACK
  - Occasionally try the other antenna to explore the channel between all four channel pairs



Frequency Diversity

- Spread signal over multiple frequencies/broader frequency band
  - For example, spread spectrum

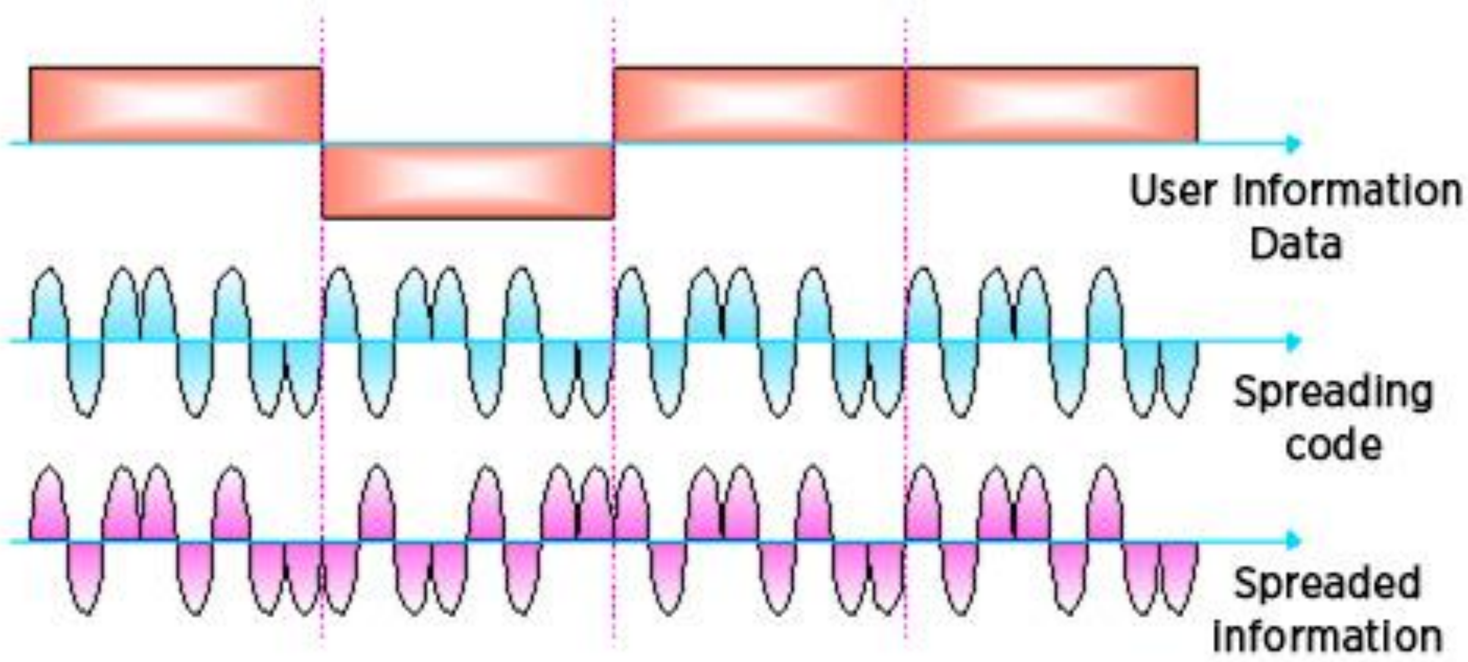


Spread Spectrum

- Spread transmission over a wider bandwidth
  - Don't put all your eggs in one basket!
  - Good for military
    - Jamming and interception becomes harder
  - Also useful to minimize impact of a "bad" frequency in regular environments
- What can be gained from this apparent waste of spectrum?
  - Immunity from various kinds of noise and multipath distortion
  - Can be used for hiding and encrypting signals
  - Several users can independently use the same higher bandwidth with very little interference

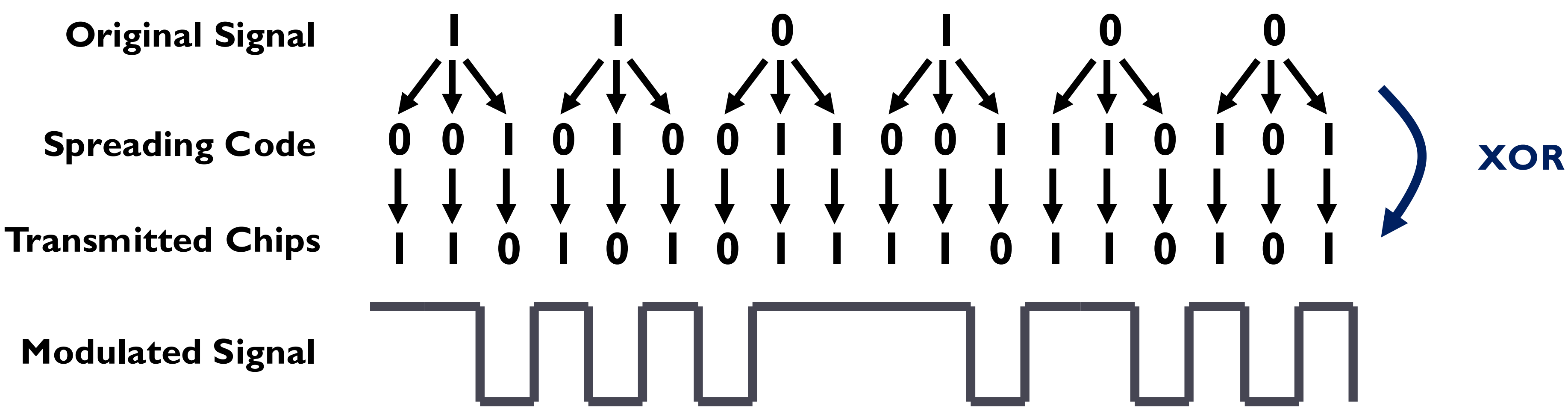
Direct Sequence Spread Spectrum (DSSS)

- Each bit in original signal is represented by multiple bits (chips) in the transmitted signal

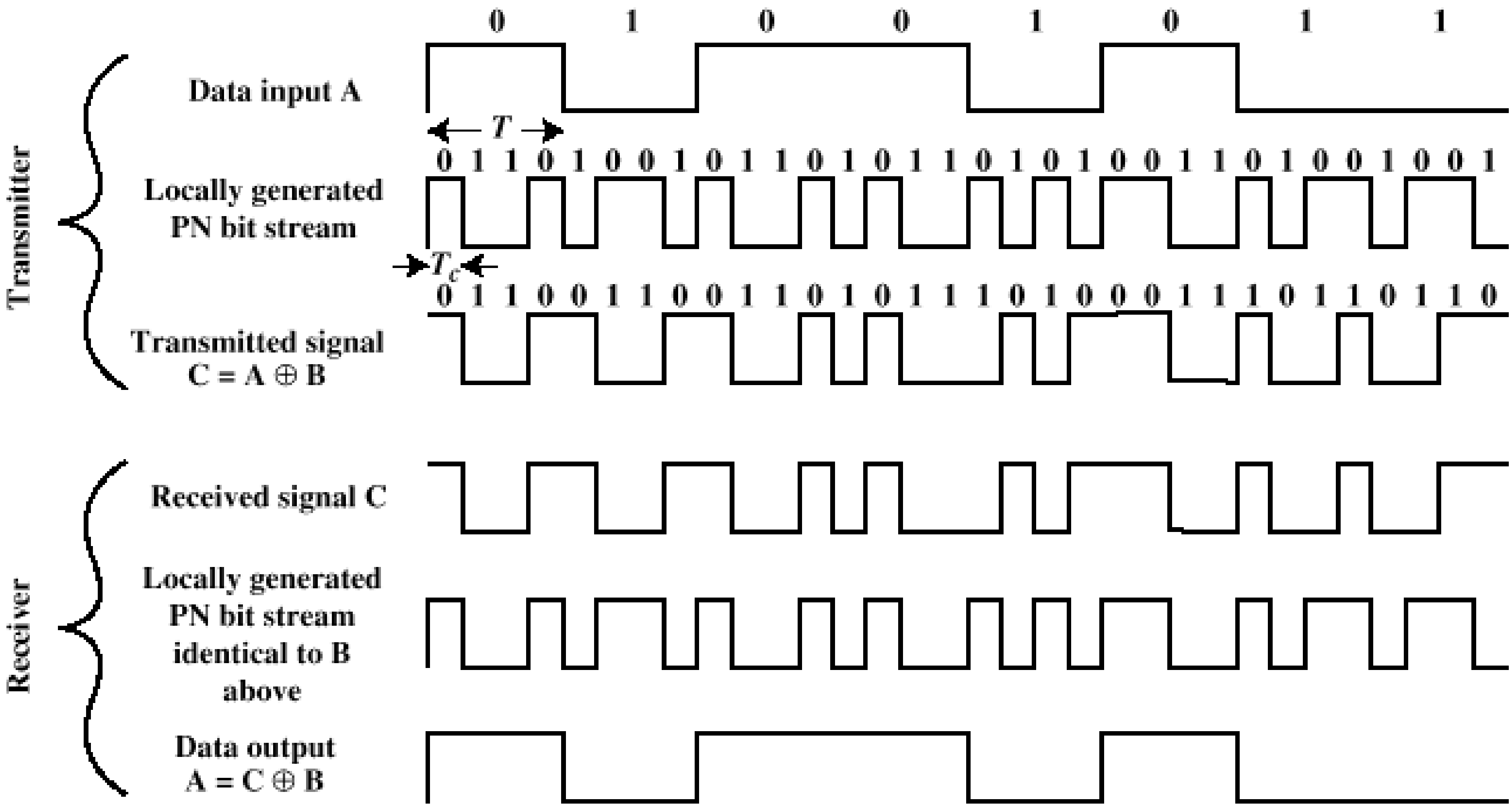


Direct Sequence Spread Spectrum (DSSS)

- Spreading code spreads signal across a wider frequency band
  - Spread is in direct proportion to number of bits used
  - e.g. exclusive-OR of the bits with the spreading code
- The resulting bit stream is used to modulate the signal



Direct Sequence Spread Spectrum (DSSS)



Properties

- Each bit is sent as multiple chips
  - Need more bps bandwidth to send signal
  - Number of chips per bit = spreading ratio
    - This is the spreading part of spread spectrum
- Need more spectral bandwidth
  - Nyquist and Shannon say so!
- Advantages
  - Transmission is more resilient.
    - DSSS signal will look like noise in a narrow band
    - Can lose some chips in a word and recover easily
  - Multiple users can share bandwidth

Example: Original 802.11 Standard (DSSS)

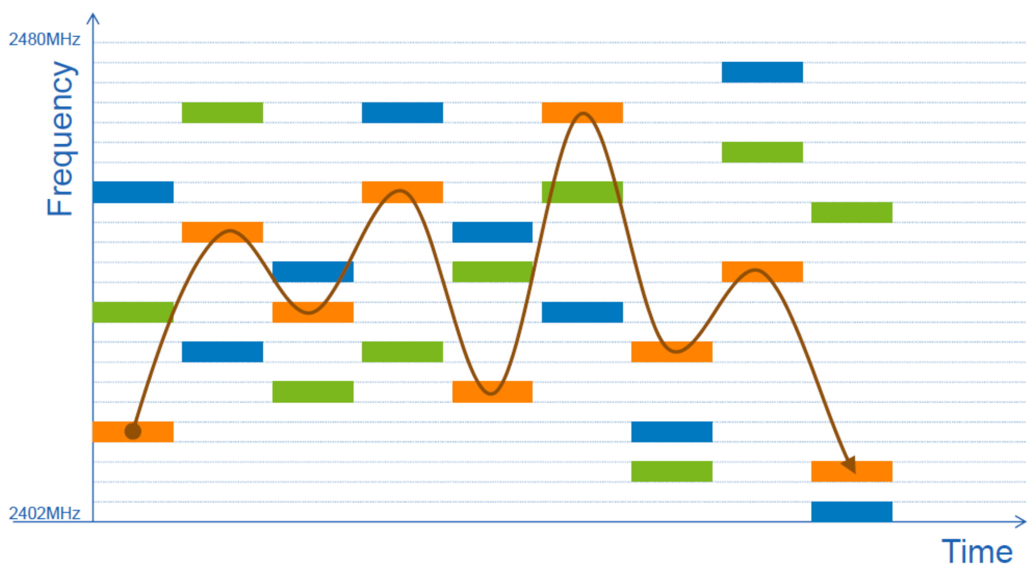
- DSSS PHY
  - 1 Msymbol/s rate
  - 11-to-1 spreading ratio
  - Barker chipping sequence
    - Barker sequence has low autocorrelation properties
      - The similarity between observations as a function of the time lag between them
  - Uses about 22 MHz
- Receiver decodes by counting the number of "1" bits in each word
  - 6 "1" bits correspond to a 0 data bit
- Data rate
  - 1 Mbps (i.e. 11 Mchips/sec)
  - Extended to 2 Mbps
    - Requires the detection of a ¼ phase shift

Example: 802.11b

- (Maximum) data rate
  - 11 Mbs
- Complementary Code Keying (CCK)
  - Complementary means that the code has good auto-correlation properties
    - Want nice properties to ease recovery in the presence of noise, multipath interference, ..
  - Each word is mapped onto an 8 bit chip sequence
  - Symbol rate at 1.375 MSymbols/sec, at 8 bpS = 11 Mbps
- Symbol rate
  - 1.375 MSymbols/sec, at 8 bpS = 11 Mbps

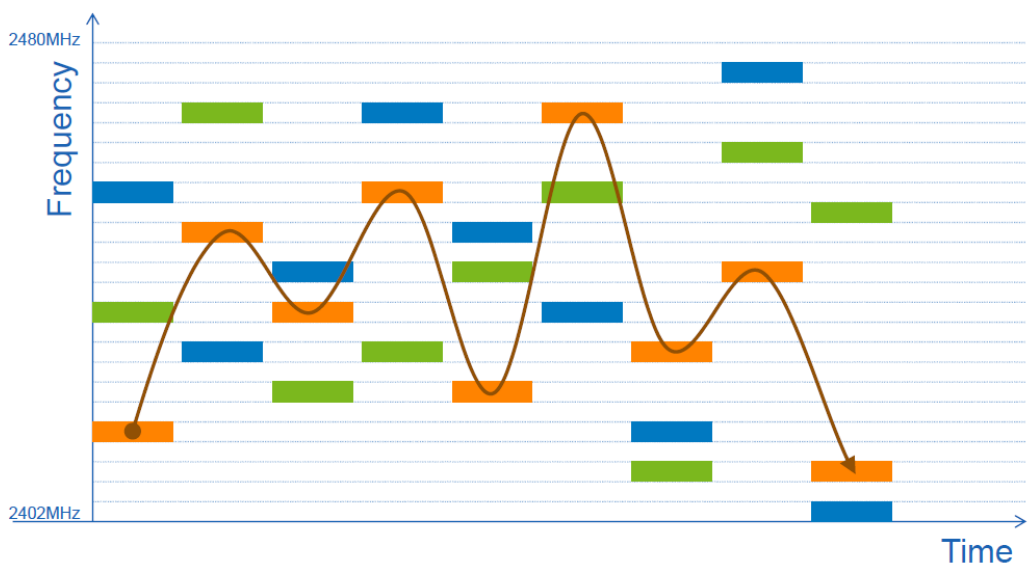
Frequency Hopping Spread Spectrum (FHSS)

- Have the transmitter hop between a seemingly random sequence of frequencies
  - Each frequency has the bandwidth of the original signal



Frequency Hopping Spread Spectrum (FHSS)

- Dwell time is the time spent using one frequency
- Spreading code determines the hopping sequence
  - Must be shared by sender and receiver (e.g. standardized)



Frequency Hopping Spread Spectrum (FHSS)

- Dwell time is the time spent using one frequency
- Spre… …nce
  - Mu…
    sta…

- Frequency hopping had many inventors
  - 1942: actress Hedy Lamarr and composer George Antheil patented Secret Communications System

  - …ll to change between 88 frequencies, and …nded to make radio-guided torpedoes …or enemies to detect or to jam
  - …ent was rediscovered in the 1950s during …earches when private companies independently developed Code Division Multiple Access, a civilian form of spread-spectrum

Example: Original 802.11 Standard (FH)

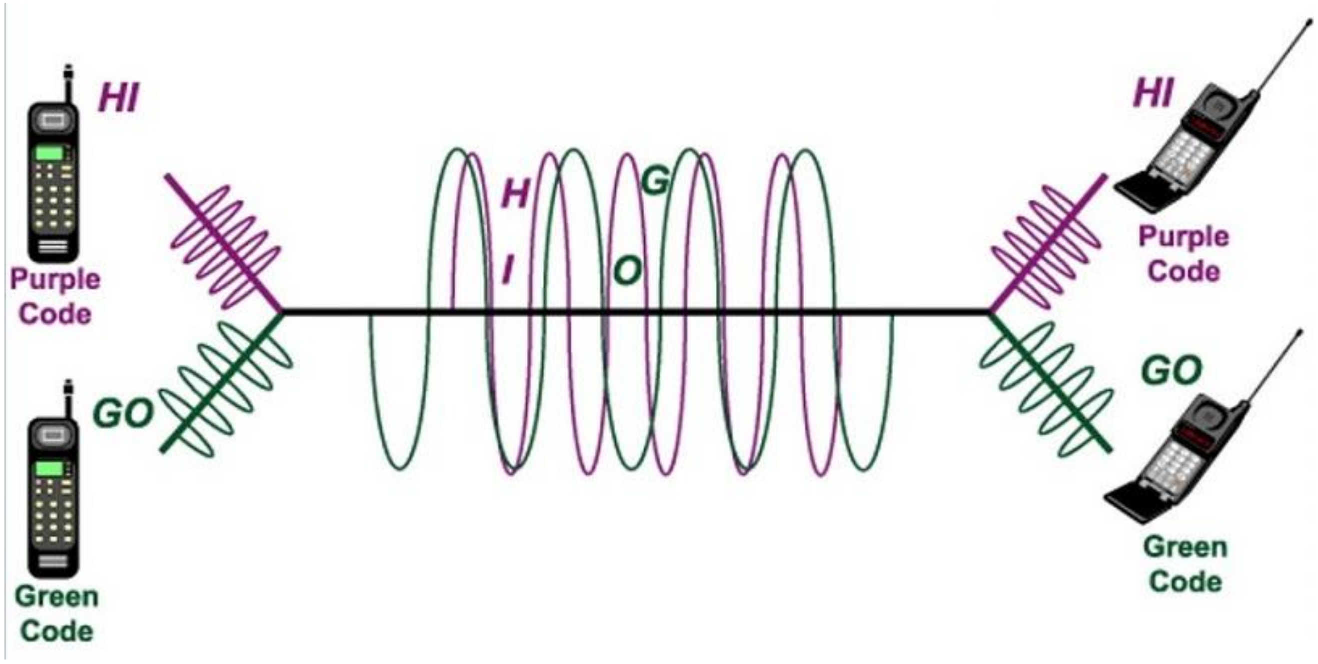
- 79 channels of 1 MHz
  - Used in North America and most of Europe
    - Other countries used different numbers
  - Each channel carried only ~1% of the bandwidth
  - 1 or 2 Mbps per channel
- Dwell time was configurable
  - FCC set an upper bound of 400 msec
  - Transmitter/receiver must be synchronized
- Standard defined 26 orthogonal hop sequences
  - Transmitter used a beacon on fixed frequency to inform the receiver of its hop sequence
- Can support multiple simultaneous transmissions – use different hop sequences
  - e.g. up to 10 co-located APs with their clients

Example: Bluetooth

- 79 frequencies with a spacing of 1 MHz
  - Other countries use different numbers of frequencies
- Frequency hopping rate is 1600 hops/s
- Maximum data rate is 1 Mbps

Code Division Multiple Access

- Users share spectrum and time, but use different codes to spread their data over frequencies



Code Division Multiple Access

- Users share spectrum and time, but use different codes to spread their data over frequencies
  - DSSS where users use different spreading sequences
  - Use spreading sequences that are orthogonal, i.e. they have minimal overlap
  - Frequency hopping with different hop sequences
- The idea is that users will only rarely overlap and the inherent robustness of DSSS will allow users to recover if there is a conflict
  - Overlap = use the same the frequency at the same time
  - The signal of other users will appear as noise

CDMA Principle

- Basic Principles of CDMA
  - D = rate of data signal
  - Break each bit into k chips - user-specific fixed pattern
  - Chip data rate of new channel = kD
- If k=6 and code is a sequence of 1s and -1s
  - For a '1' bit, A sends code as chip pattern
    - <c1, c2, c3, c4, c5, c6>
  - For a '0' bit, A sends complement of code
    - <-c1, -c2, -c3, -c4, -c5, -c6>
- Receiver knows sender's code and performs electronic decode function

$$S_u(d) = d1 \times c1 + d2 \times c2 + d3 \times c3 + d4 \times c4 + d5 \times c5 + d6 \times c6$$

  - <d1, d2, d3, d4, d5, d6> = received chip pattern
  - <c1, c2, c3, c4, c5, c6> = sender's code

CDMA Example

- User A code = <1, –1, –1, 1, –1, 1>
  - To send a 1 bit = <1, –1, –1, 1, –1, 1>
  - To send a 0 bit = <–1, 1, 1, –1, 1, –1>
- User B code = <1, 1, –1, – 1, 1, 1>
  - To send a 1 bit = <1, 1, –1, –1, 1, 1>
- Receiver receiving with A's code
  - (A's code) x (received chip pattern)
    - User A '1' bit: 6 -> 1
    - User A '0' bit: -6 -> 0
    - User B '1' bit: 0 -> unwanted signal ignored

CDMA Example

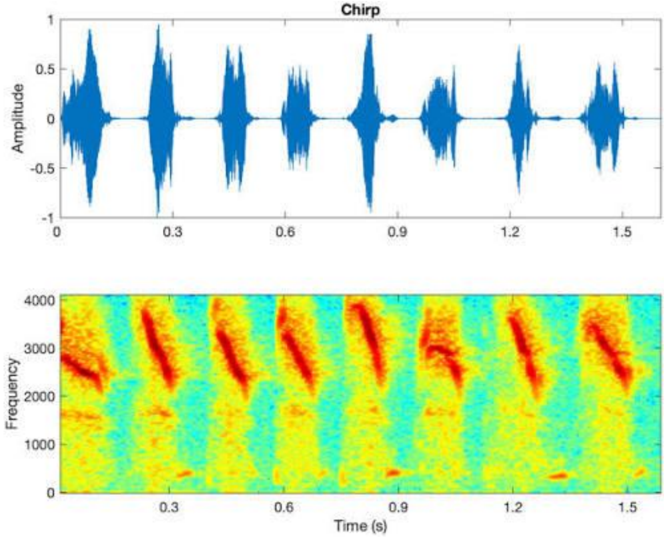
- CDMA cellular standard
  - Used in the US, e.g. Sprint
- Allocates 1.228 MHz for base station to mobile communication
  - Shared by 64 "code channels"
  - Used for voice (55), paging service (8), and control (1)
- Provides a lot error coding to recover from errors
  - Voice data is 8550 bps
  - Coding and FEC increase this to 19.2 kbps
  - Then spread out over 1.228 MHz using DSSS; uses QPSK

Discussion

- Spread spectrum is very widely used
- Effective against noise and multipath
  - Signal looks like noise to other nodes
  - Multiple transmitters can use the same frequency range
- FCC requires the use of spread spectrum in ISM band
  - If signal is above a certain power level
- Is also used in higher speed 802.11 versions.
  - No surprise!

Chirp Spread Spectrum

- Chirp is signal that continuously sweeps frequency with time



Chirp Spread Spectrum

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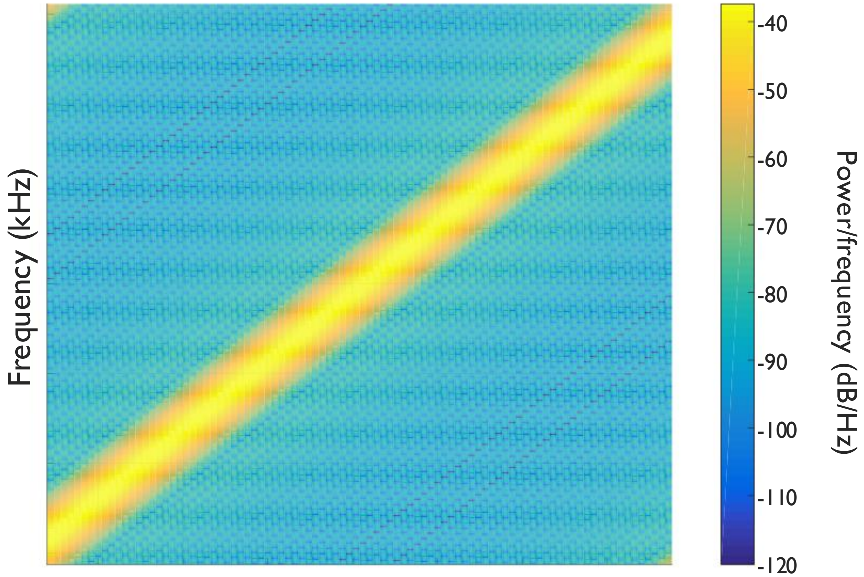
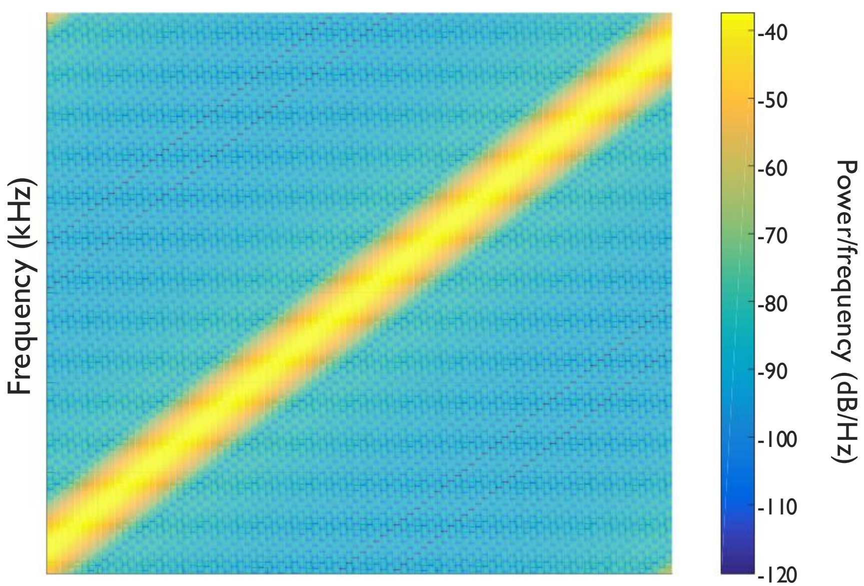


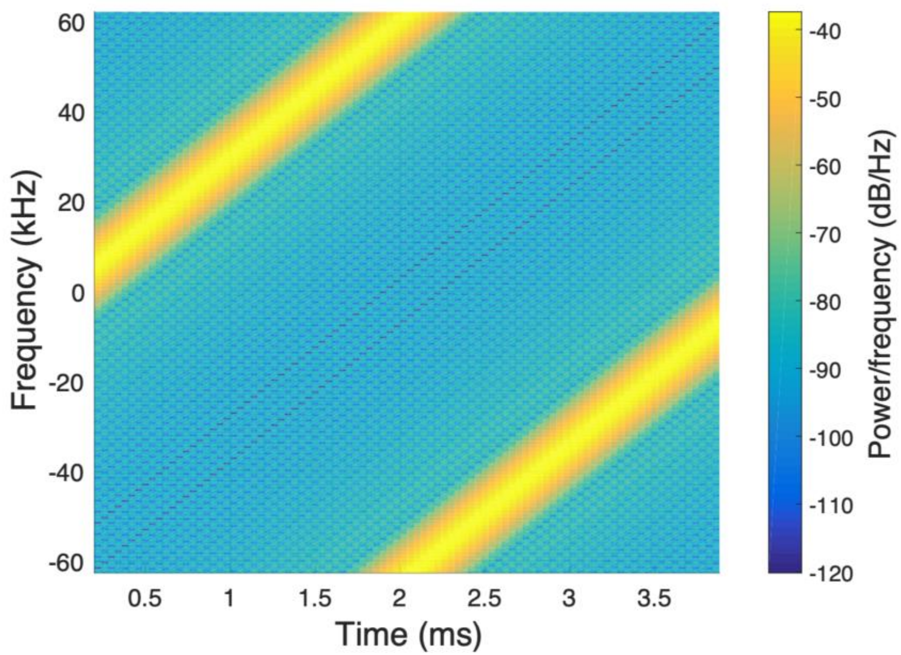
LoRa

- LoRa encoding
  - Modulate starting frequency to create symbols

bit = '0'

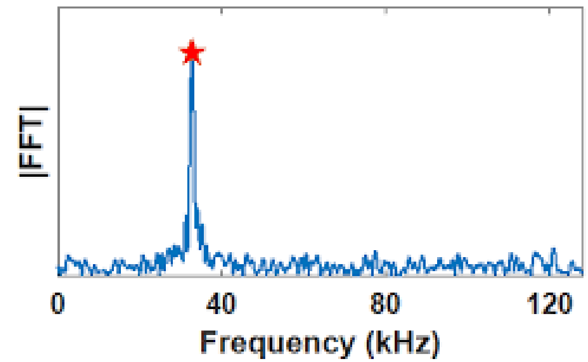


bit = '1'



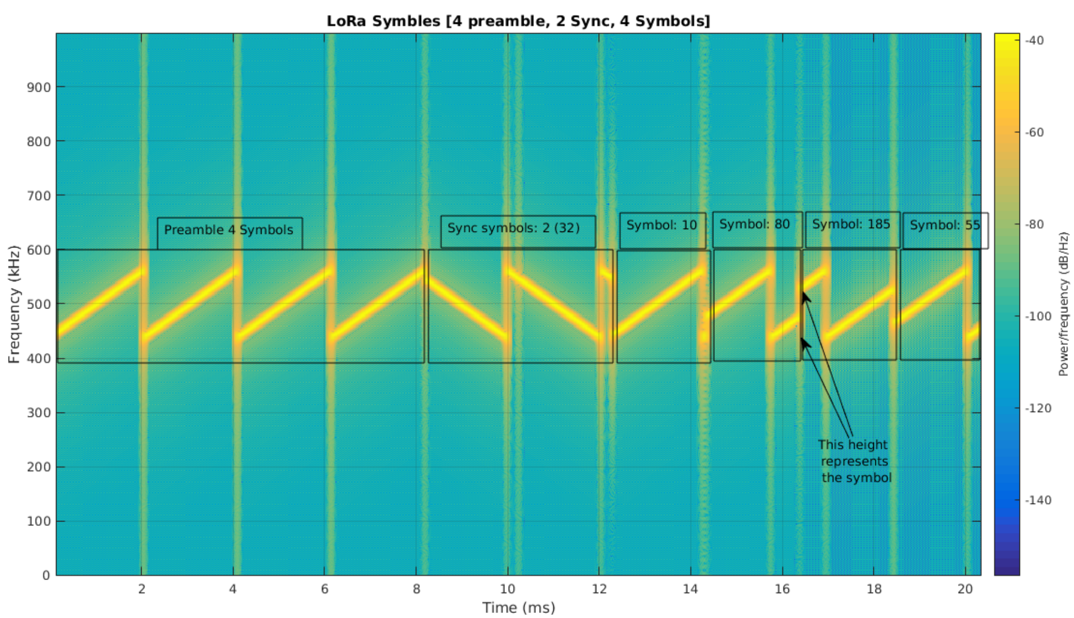
LoRa

- LoRa decoding
  - The receiver puts the signal through an FFT
  - Peak in FFT domain represents the start frequency for the transmission



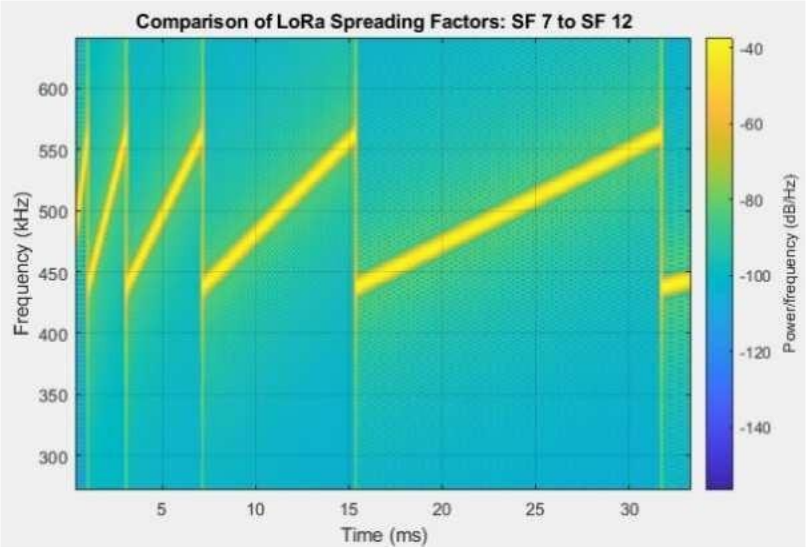
LoRa

- **LoRa packet transmission**
  - Preamble and sync: set of up-chirps
  - Start of packet preceded by a set of down-chirps
  - Packet data transmitted using cyclic shift



LoRa

- Spreading factor
  - Length of chirp
  - Short SF
    - Faster to decode
    - More susceptible to interference
  - Long SF
    - Limits data rate
    - More robust



LoRa: Low-Power Wide-Area Spread Spectrum

- LoRa
  - Range: 1–10 km
  - Battery life: 5–10 years
  - Cost: $10–$20 per node
  - Data rate: only 100 bps – 250 kbps in exchange for that range and battery life

Time Diversity

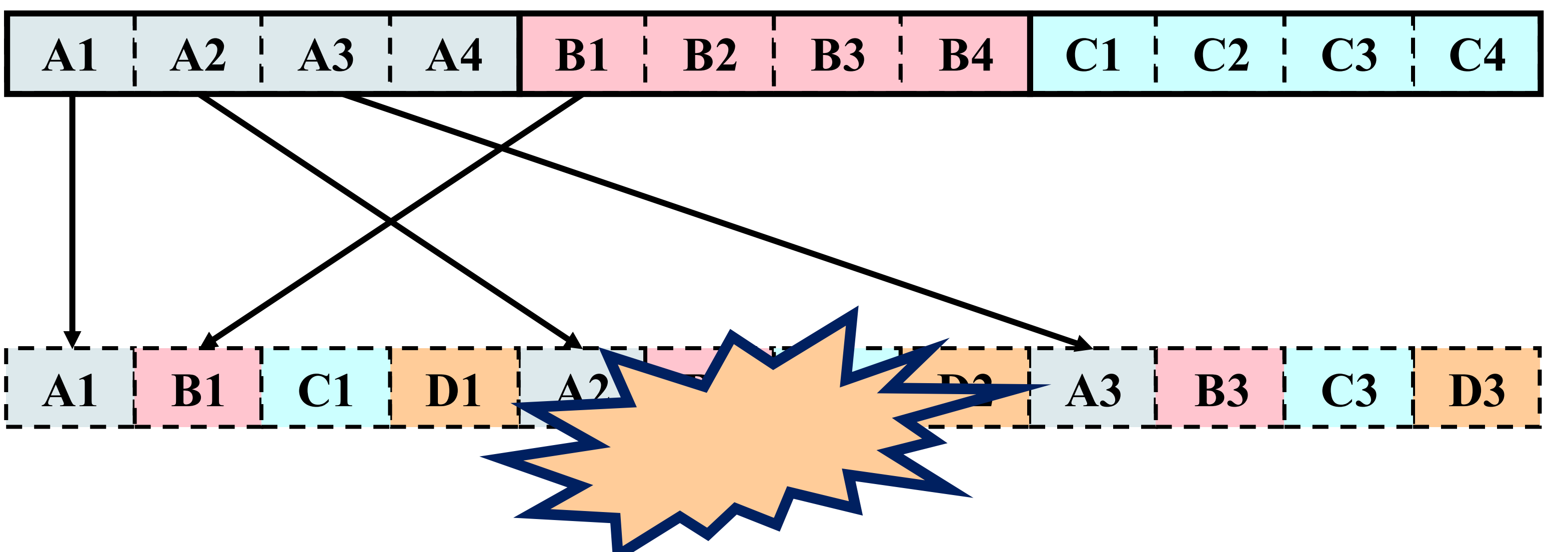
- Spread data out over time
- Expand bit stream into a richer digital signal
  - Useful for bursty errors, e.g. slow fading
  - A specific form of channel coding

Time Redundancy: Bit Stream Level

- Protect digital data by introducing redundancy in the transmitted data
  - Error detection codes: can identify certain types of errors
  - Error correction codes: can fix certain types of errors
- Block codes provide Forward Error Correction (FEC) for blocks of data
  - (n, k) code: n bits are transmitted for k information bits
  - Simplest example: parity codes
  - Many different codes exist: Hamming, cyclic, Reed-Solomon, …
- Convolutional codes provide protection for a continuous stream of bits
  - Coding gain is n/k
  - Turbo codes: convolutional code with channel estimation

Time Diversity Example

- Spread blocks of bytes out over time
- Can use FEC or other error recovery techniques to deal with burst errors



Error Detection/Recovery

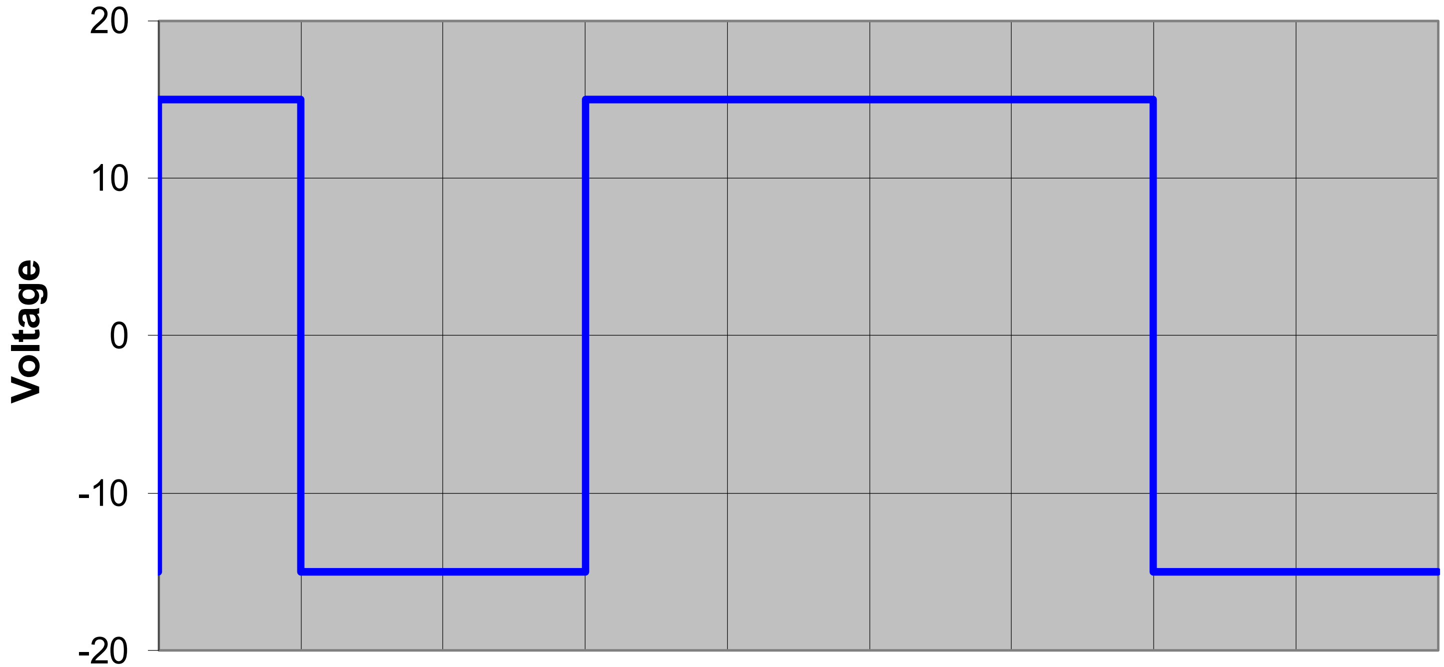
- Adds redundant information that checks for errors
  - And potentially fix them
  - If not, discard packet and resend
- Occurs at many levels
  - Demodulation of signals into symbols (analog)
  - Bit error detection/correction (digital)—our main focus
    - Within network adapter (CRC check)

Error Detection/Recovery

- Analog Errors
  - Example of signal distortion
- Hamming distance
  - Parity and voting
  - Hamming codes
- Error bits or error bursts?
- Digital error detection
  - Two-dimensional parity
  - Cyclic Redundancy Check (CRC)

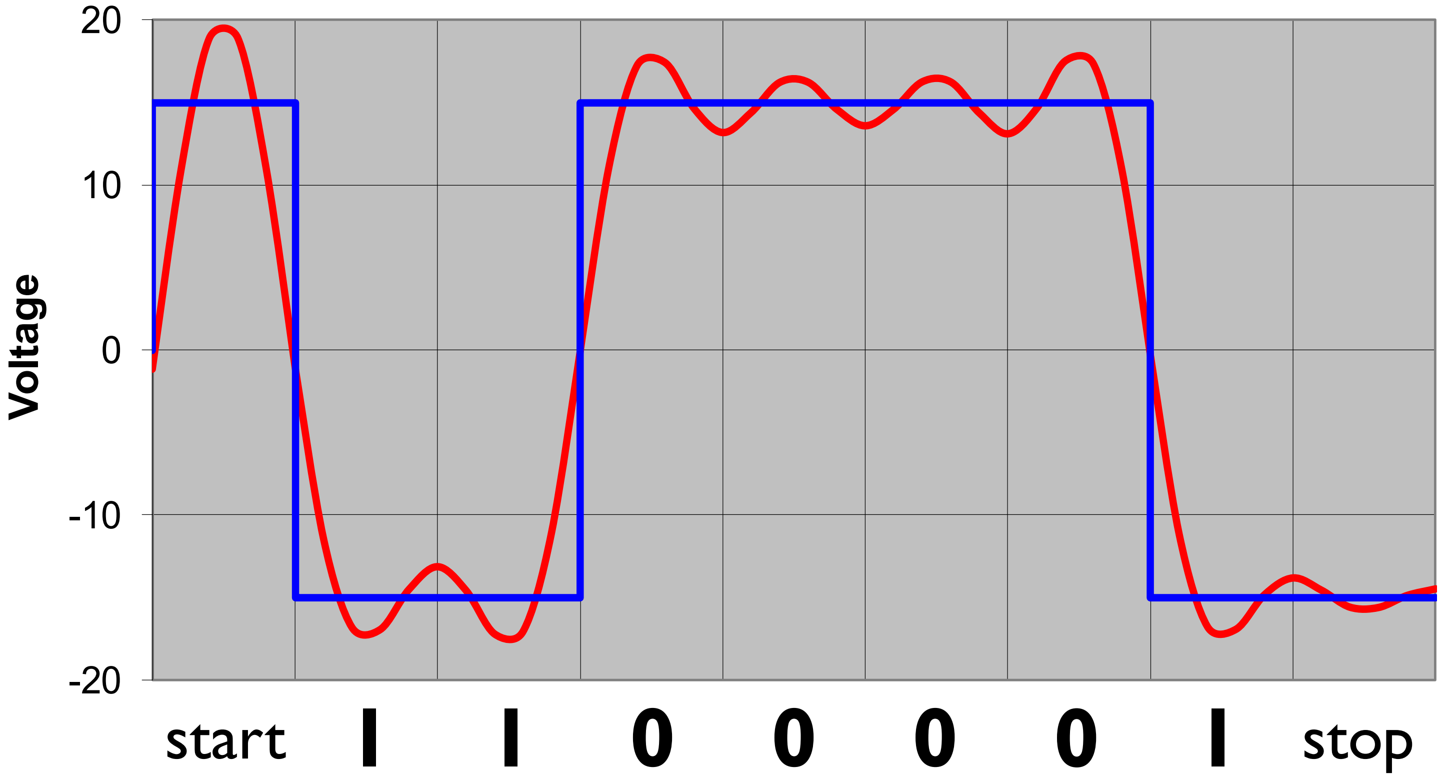
Analog Errors

- Consider the following encoding of ‘Q’



start 1 1 0 0 0 0 1 stop

Encoding isn't perfect



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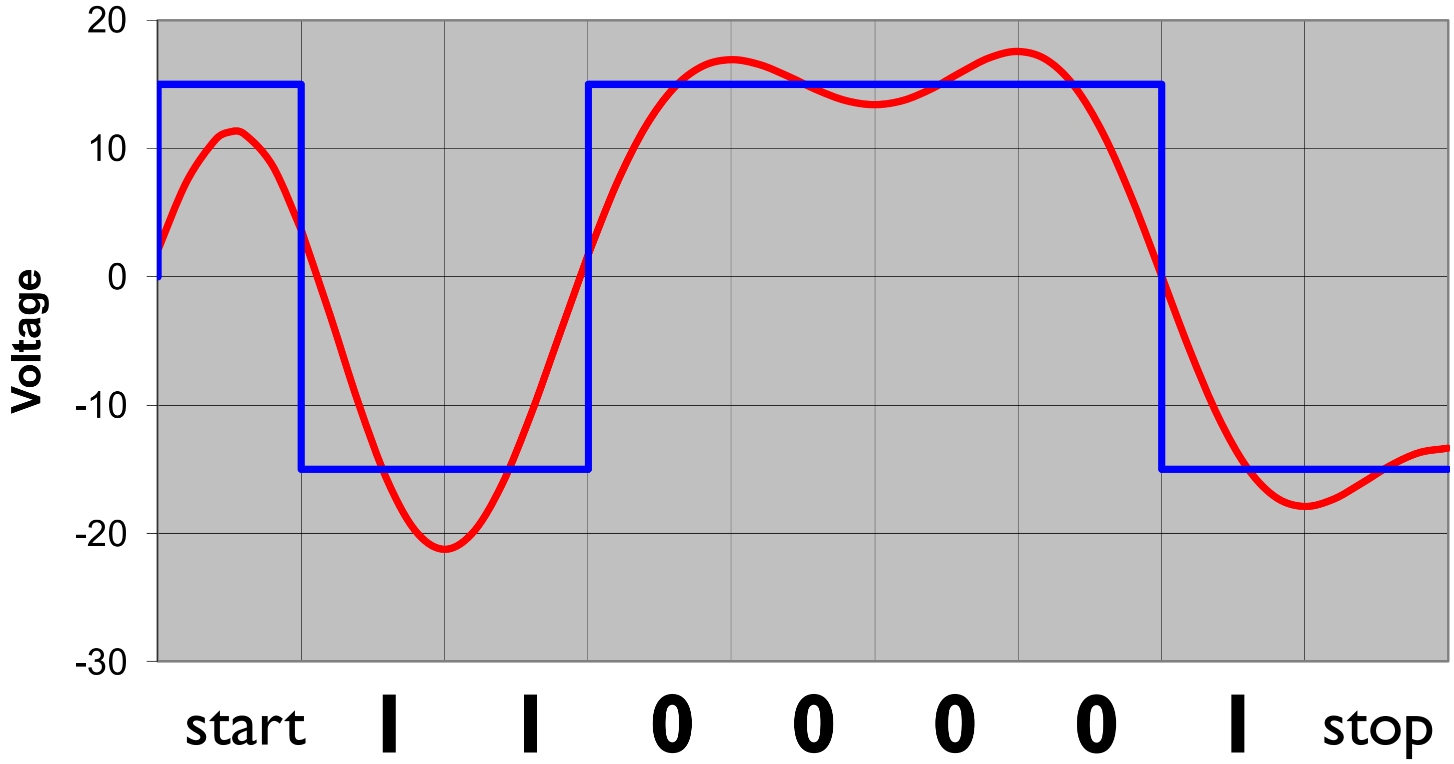
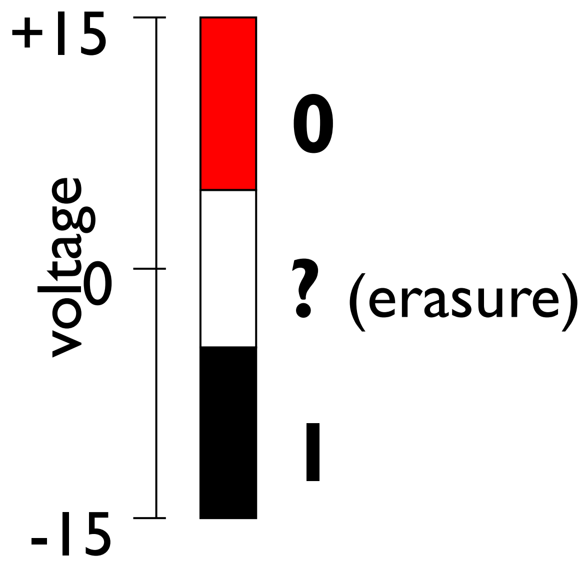
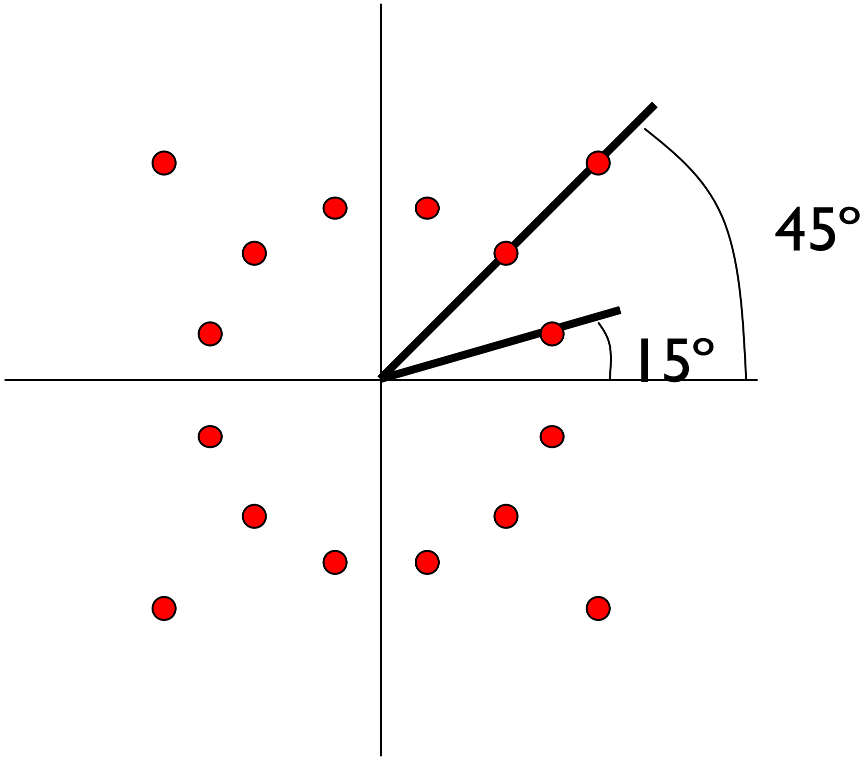


Symbols



possible binary voltage encoding symbol neighborhoods and erasure region

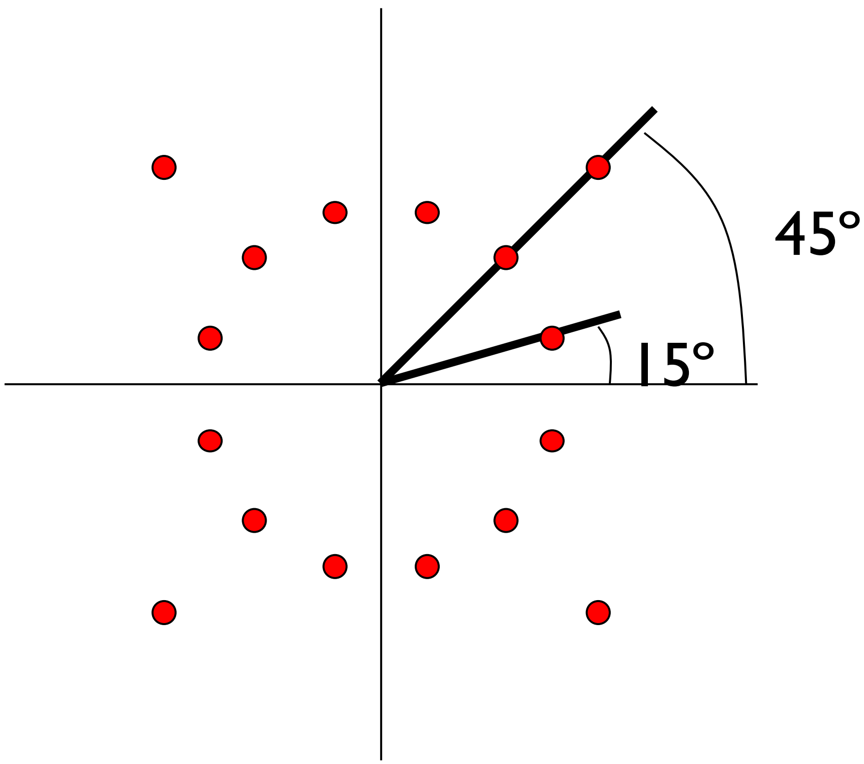
Symbols



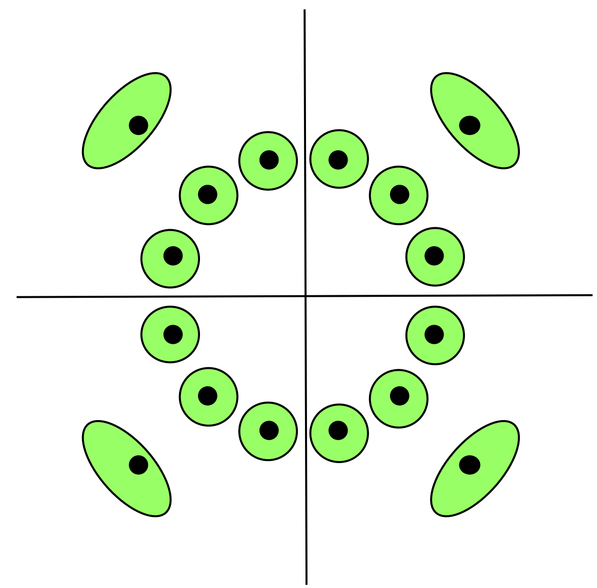
16-symbol example

- QAM
  - Phase and amplitude modulation
- 2-dimensional representation
  - Angle is phase shift
  - Radial distance is new amplitude

Symbols



16-symbol example



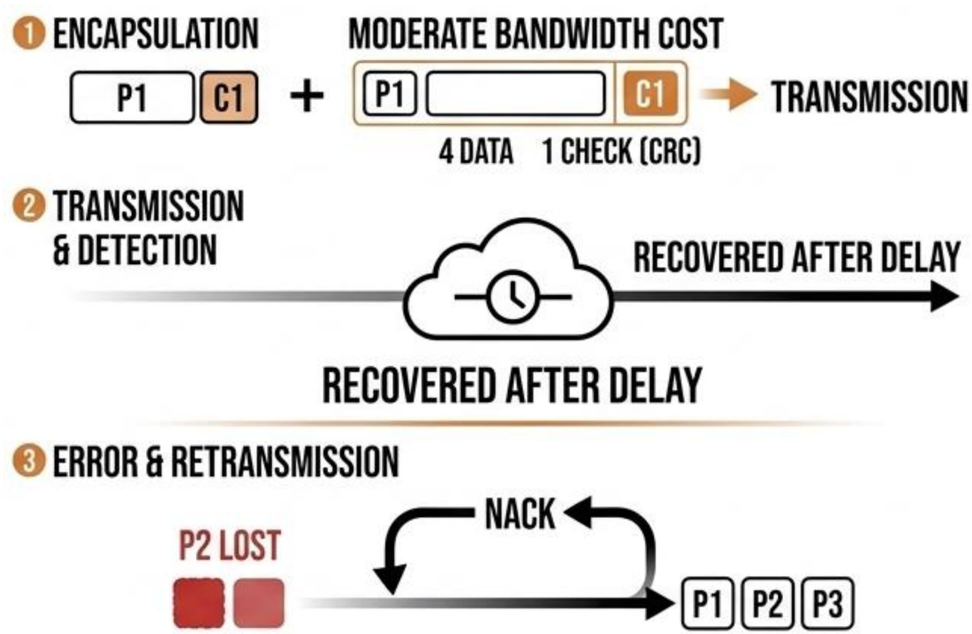
possible QAM symbol neighborhoods in green; all other space results in erasure

Digital error detection and correction

- Input: decoded symbols
  - Some correct
  - Some incorrect
  - Some erased
- Output:
  - Correct blocks (or codewords, or frames, or packets)
  - Erased blocks

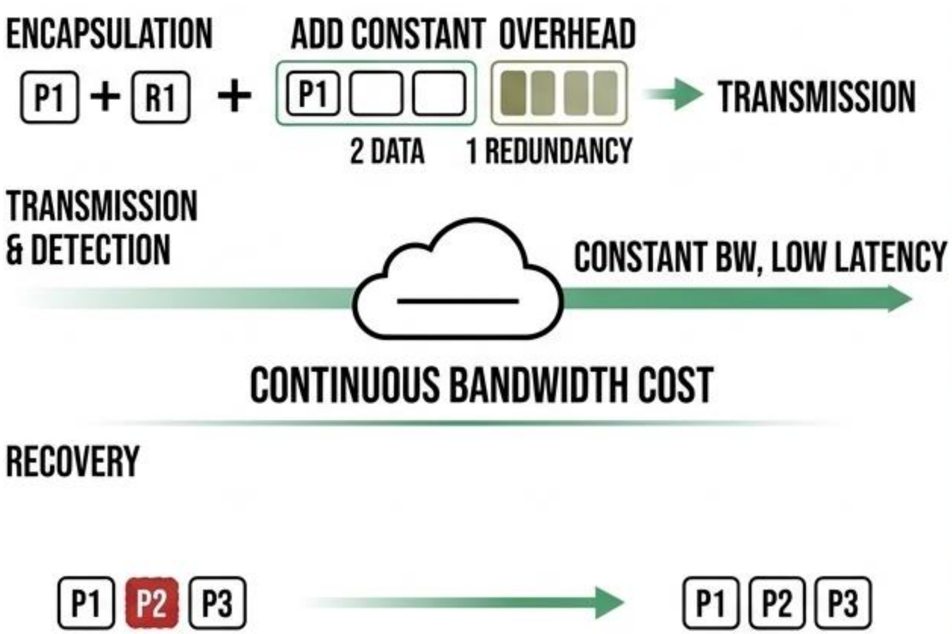
Error Detection

- Add some check bits to message
  - Use check bits to determine correctness

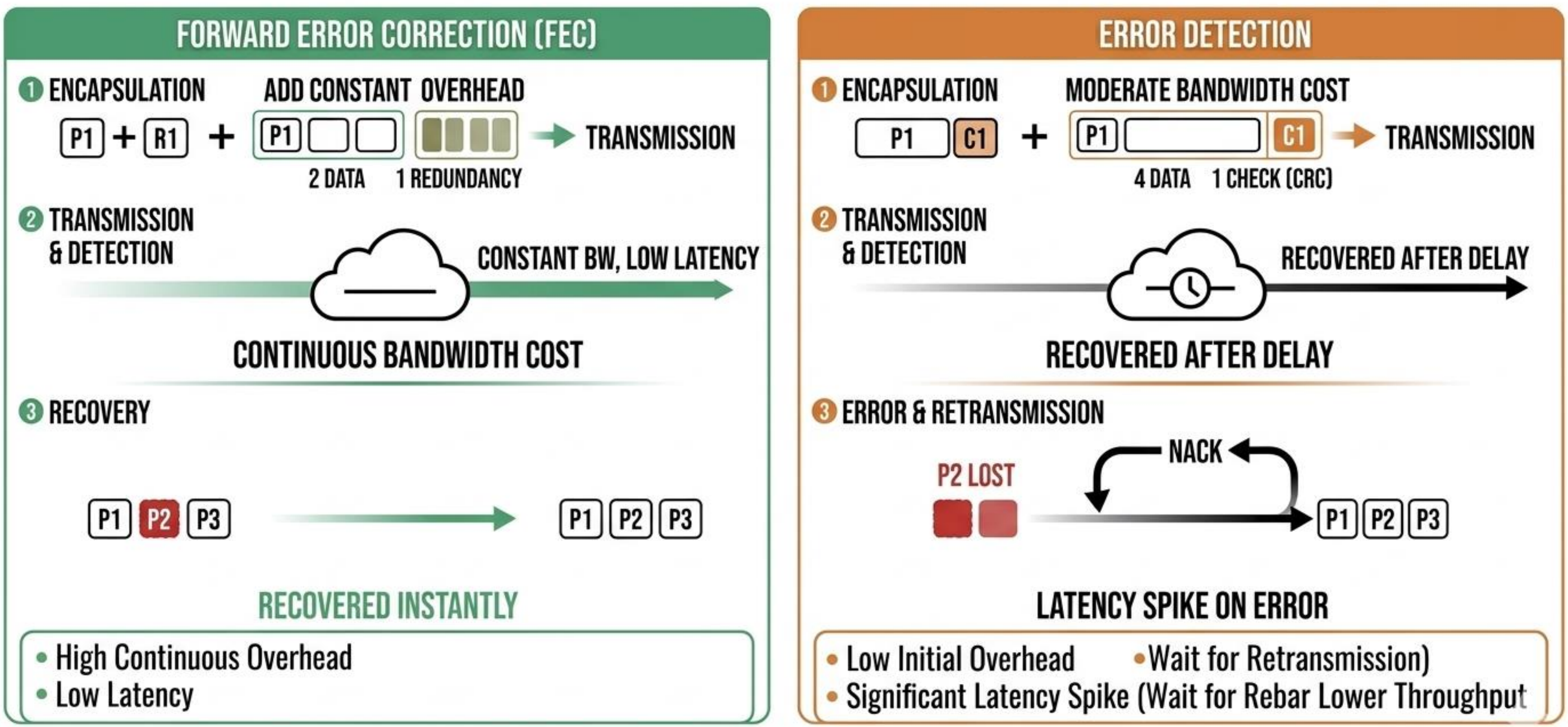


Forward Error Correction (FEC)

- Add enough redundancy for the receiver to fix errors
  - No retransmission request needed
  - FEC spends bandwidth on every message, even error-free ones



Error Correction vs. Error Detection



Error Detection Probabilities

- Definitions
  - P_b : Probability of single bit error (BER)
  - P_1 : Probability that a frame arrives with no bit errors
  - P_2 : While using error detection, the probability that a frame arrives with one or more undetected errors
  - P_3 : While using error detection, the probability that a frame arrives with one or more detected bit errors but no undetected bit errors

Error Detection Probabilities

- With no error detection

Single bit error

No bit errors $$P_1 = (1 - P_b)^F$$

Undetected errors $$P_2 = 1 - P_1$$

Detected errors $$P_3 = 0$$

  - F = Number of bits per frame

Error Detection Process

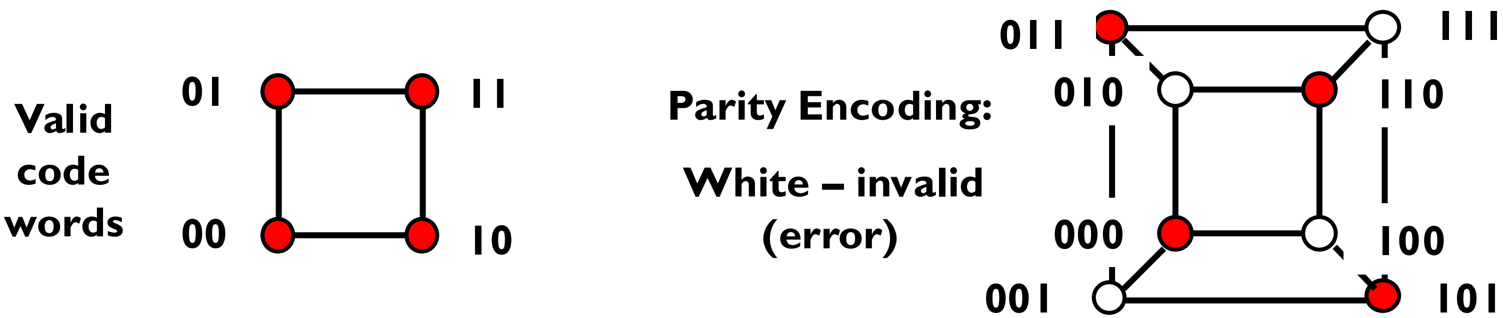
- Transmitter
  - For a given frame, an error-detecting code (check bits) is calculated from data bits
  - Check bits are appended to data bits
- Receiver
  - Separates incoming frame into data bits and check bits
  - Calculates check bits from received data bits
  - Compares calculated check bits against received check bits
  - Detected error occurs if mismatch

Parity

- Parity bit appended to a block of data
- Even parity
  - Added bit ensures an even number of 1s
- Odd parity
  - Added bit ensures an odd number of 1s
- Example
  - 7-bit character 1110001
  - Even parity 1110001 0
  - Odd parity 1110001 1

Parity: Detecting Bit Flips

- 1-bit error detection with parity
  - Add an extra bit to a code to ensure an even (odd) number of 1s
  - Every code word has an even (odd) number of 1s



Voting: Correcting Bit Flips

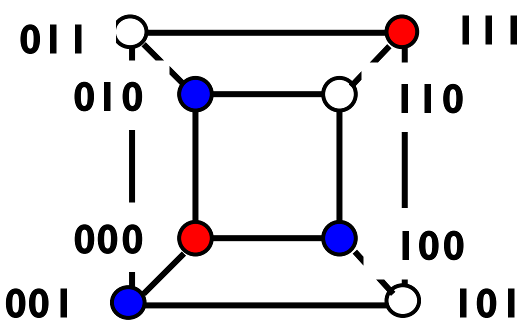
- 1-bit error correction with voting
  - Every codeword is transmitted n times
  - Codeword is 3 bits long

Valid code words

Voting:

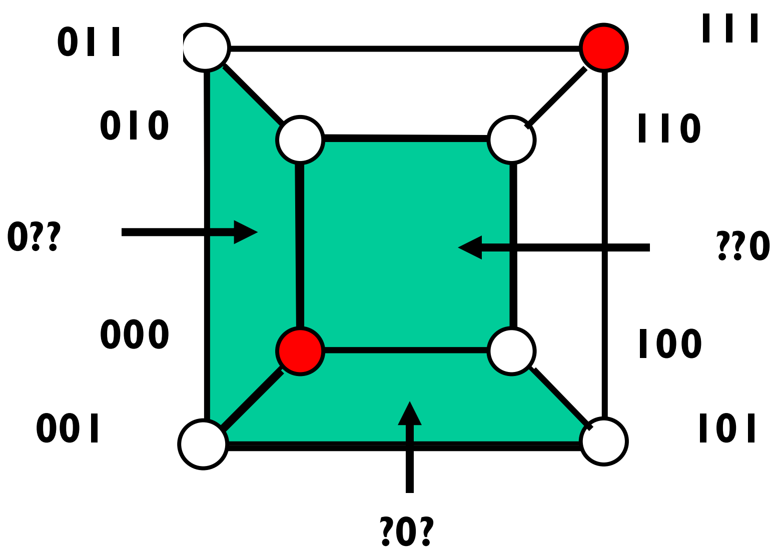
White – correct to 1

Blue - correct to 0



Voting: 2-bit Erasure Correction

- Every code word is copied 3 times



2-erasure planes in green
remaining bit not ambiguous

cannot correct 1-error and 1-erasure

Hamming Distance

- The Hamming distance between two code words is the minimum number of bit flips to move from one to the other
  - Example:
  - 00101 and 00010
  - Hamming distance of 3

Minimum Hamming Distance

- The minimum Hamming distance of a code is the minimum distance over all pairs of codewords
  - Minimum Hamming Distance for parity
    - 2
  - Minimum Hamming Distance for voting
    - 3

Coverage

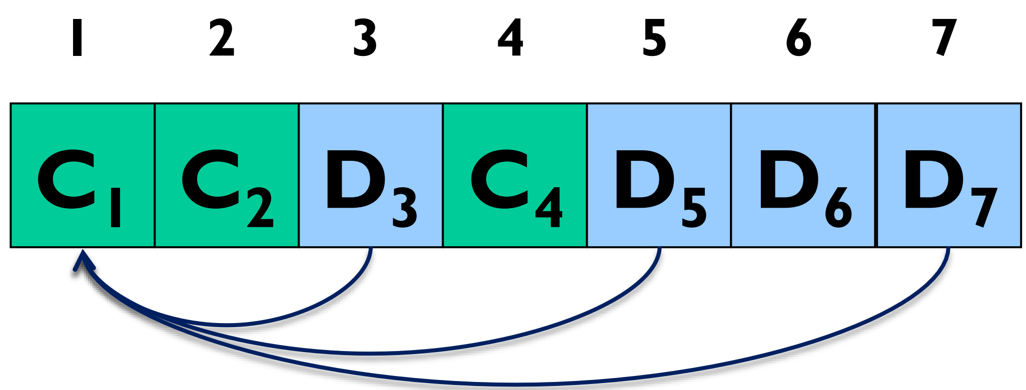
- N-bit error detection
  - No code word changed into another code word
  - Requires Hamming distance of N+1
- N-bit error correction
  - N-bit neighborhood: all codewords within N bit flips
  - No overlap between N-bit neighborhoods
  - Requires hamming distance of 2N+1

Hamming Codes

- Linear error-correcting code
- Named after Richard Hamming
- Simple, commonly used in RAM (e.g., ECC-RAM)
- Can detect up to 2-bit errors
- Can correct up to 1-bit errors

Hamming Codes

- Construction
  - number bits from 1 upward
  - powers of 2 are check bits
  - all others are data bits
  - Check bit j: XOR of all k for which (j AND k) = j
    - Example:
      - 4 bits of data, 3 check bits



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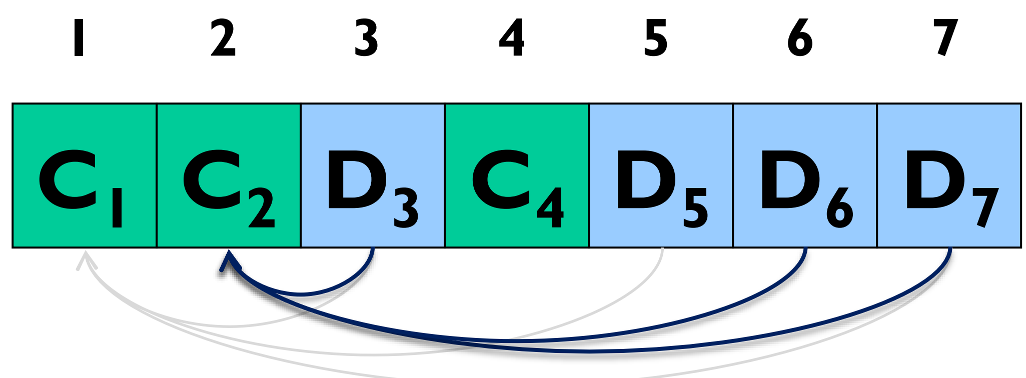

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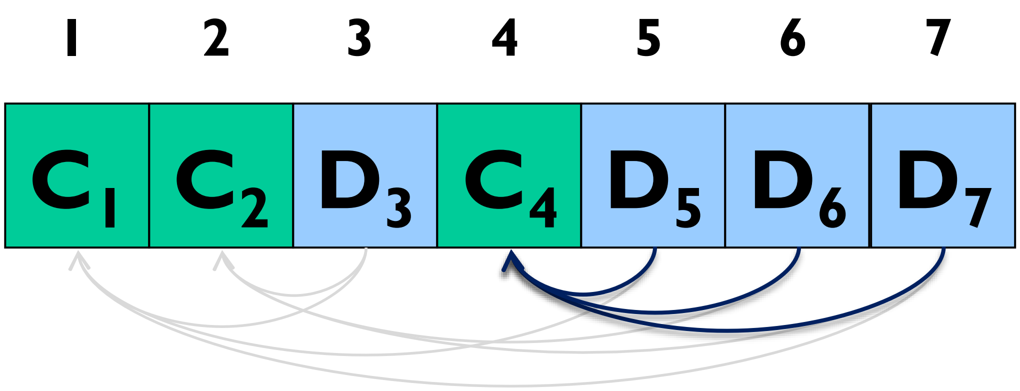

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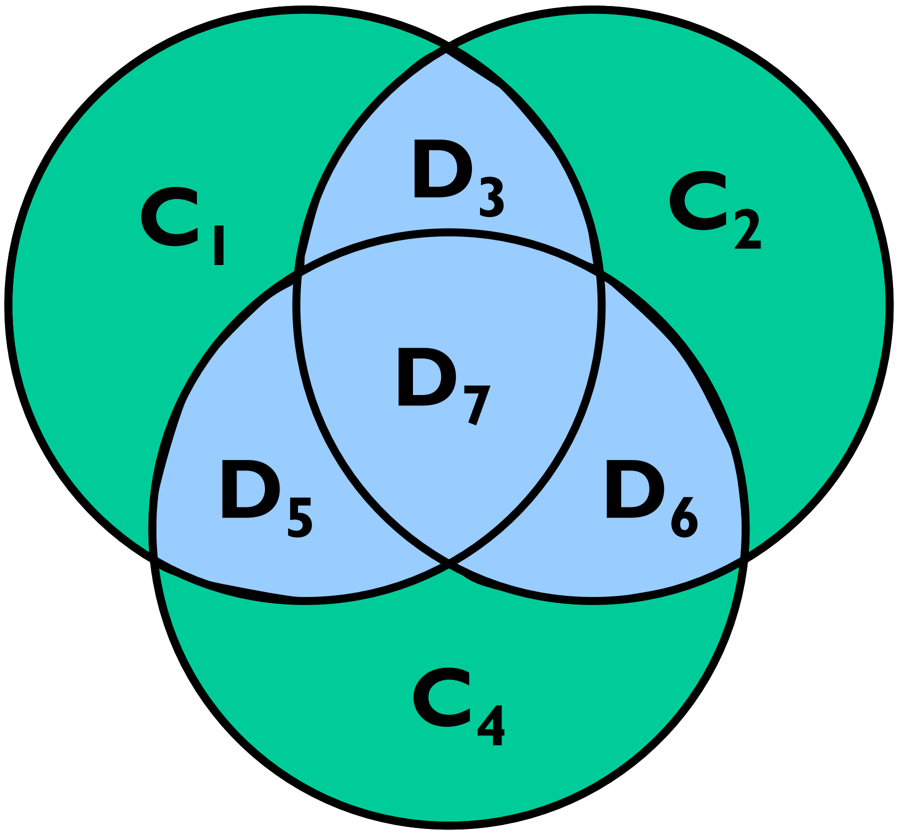


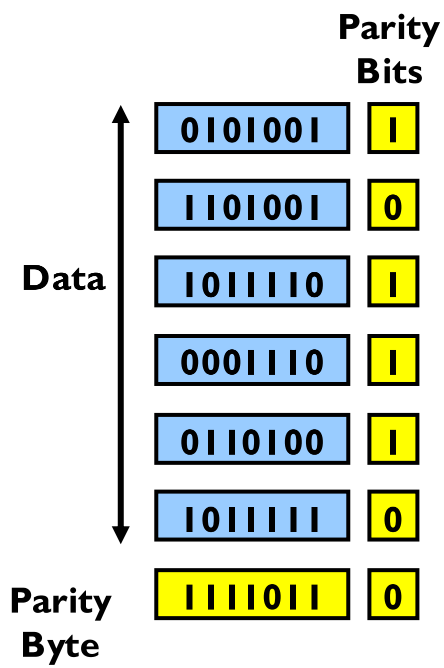
What are we trying to handle?

- Worst case errors
  - We solved this for 1 bit error
  - Can generalize, but will get expensive for more bit errors
- Probability of error per bit
  - Flip each bit with some probability, independently of others
- Burst model
  - Probability of back-to-back bit errors
  - Error probability dependent on adjacent bits
  - Value of errors may have structure
- Why assume bursts?
  - Appropriate for some media (e.g., radio)
  - Faster signaling rate enhances such phenomena

Digital Error Detection Techniques

- Two-dimensional parity
  - Detects up to 3-bit errors
  - Good for burst errors
- IP checksum
  - Simple addition
  - Simple in software
  - Used as backup to CRC
- Cyclic Redundancy Check (CRC)
  - Powerful mathematics
  - Tricky in software, simple in hardware
  - Used in network adapter

Two-Dimensional Parity



- Use 1-dimensional parity
  - Add one bit to a 7-bit code to ensure an even/odd number of 1s
- Add 2nd dimension
  - Add an extra byte to frame
    - Bits are set to ensure even/odd number of 1s in that position across all bytes in frame
- Comments
  - Catches all 1-, 2- and 3-bit and most 4-bit errors

Two-Dimensional Parity

0	1	0	0	0	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	0	1	0	0	1
0	0	1	0	0	0	1	1	1

What happens if…

Can detect exactly which bit flipped
Can also correct it!

0	1	~~0~~ 1	0	0	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	0	1	0	0	1
0	0	1	0	0	0	1	1	1

What about 2-bit errors?

Can detect the two-bit error

Can’t detect a problem here

Can’t tell which bits are flipped, so can’t correct

0	1	~~0~~ 1	0	0	~~1~~ 0	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	0	1	0	0	1
0	0	1	0	0	0	1	1	1

What about 2-bit errors?

Could be the dotted pair or the dashed pair.
Can't correct 2-bit error.

If these four parity bits don't match
Which bits could be in error?

0	1	0	0	0	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	0	1	0	0	1
0	0	1	0	0	0	1	1	1

What about 3-bit errors?

Can detect the three-bit error

0	1	~~0~~ 1	0	0	~~1~~ 0	1	~~1~~ 0	0
0	1	1	0	1	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	0	1	0	0	1
0	0	1	0	0	0	1	1	1

But you can't correct (eg if dashed bits got flipped instead of the dotted ones)

What about 4-bit errors?

Are there any 4-bit errors this scheme *can* detect?

0	1	0	0	0	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	1	1	1	1	0
0	1	1	0	0	1	0	0	1
0	0	1	0	0	0	1	1	1

What about 4-bit errors?

Can you think of a 4-bit error this scheme can't detect?

~~0~~ 1	1	~~0~~ 1	0	0	1	1	1	0
0	1	1	0	1	1	1	1	0
~~0~~ 1	1	~~1~~ 0	0	1	1	1	1	0
0	1	1	0	0	1	0	0	1
0	0	1	0	0	0	1	1	1

Internet Checksum

- Idea
  - Add up all the words
  - Transmit the sum
  - Use 1's complement addition on 16bit codewords
  - Example
    - Codewords: -5 -3
    - 1's complement binary: 1010 1100
    - 1's complement sum 1000
- Comments
  - Small number of redundant bits
  - Easy to implement
  - Not very robust
  - Eliminated in IPv6

IP Checksum

```
u_short cksum(u_short *buf, int count) {
  register u_long sum = 0;
  while (count--) {
       sum += *buf++;
       if (sum & 0xFFFF0000) {
       /* carry occurred, so wrap around */
              sum &= 0xFFFF;
              sum++;
       }
  }
  return ~(sum & 0xFFFF);
}
```

What could cause this check to fail?

Simplified CRC-like protocol using regular integers

- Basic idea
  - Both endpoints agree in advance on divisor value $C = 3$
  - Sender wants to send message $M = 10$
  - Sender computes X such that C divides $10M + X$
  - Sender sends codeword $W = 10M + X$
  - Receiver receives W' and checks whether C divides W'
    - If so, then probably no error
    - If not, then error

Simplified CRC-like protocol using regular integers

- Intuition
  - If C is large, it's unlikely that bits are flipped exactly to land on another multiple of C
  - CRC is vaguely like this, but uses polynomials instead of numbers

Cyclic Redundancy Check (CRC)

- Given
  - Message M = 10011010
  - Represented as Polynomial $M(x)$

    $= \mathbf{1} *x^7 + \mathbf{0} *x^6 + \mathbf{0} *x^5 + \mathbf{1} *x^4 + \mathbf{1} *x^3 + \mathbf{0} *x^2 + \mathbf{1} *x + \mathbf{0}$

    $= x^7 + x^4 + x^3 + x$
- Select a divisor polynomial $C(x)$ with degree k
  - Example with $k = 3$:
    - $C(x) = x^3 + x^2 + 1$
    - Represented as 1101
- Transmit a polynomial $P(x)$ that is evenly divisible by $C(x)$
  - $P(x) = M(x) * x^k + k$ check bits

How can we determine these k bits?

Properties of Polynomial Arithmetic

- Coefficients are modulo 2

  $(x^3 + x) + (x^2 + x + 1) = \ldots$

  $\ldots x^3 + x^2 + 1$

  $(x^3 + x) - (x^2 + x + 1) = \ldots$

  $\ldots x^3 + x^2 + 1$ also!

- Addition and subtraction are both xor!
- Need to compute R such that $C(x)$ divides $P(x) = M(x) \cdot x^k + R(x)$
- So $R(x) =$ remainder of $M(x) \cdot x^k / C(x)$
  - Will find this with polynomial long division

CRC - Sender

- Given
  - $M(x) =$ 10011010 $= x^7 + x^4 + x^3 + x$
  - $C(x) =$ 1101 $= x^3 + x^2 + 1$
- Steps
  - $T(x) = M(x) * x^k$ (add zeros to increase deg. of $M(x)$ by k)
  - Find remainder, $R(x)$, from $T(x)/C(x)$
  - $P(x) = T(x) - R(x) \Rightarrow M(x)$ followed by $R(x)$
- Example
  - $T(x) =$ 10011010000
  - $R(x) =$ 101
  - $P(x) =$ 10011010101

CRC - Receiver

- Receive Polynomial $P(x) + E(x)$
  - $E(x)$ represents errors
  - $E(x) = 0$, implies no errors
- Divide ($P(x) + E(x)$) by $C(x)$
  - If result = 0, either
    - No errors ($E(x) = 0$, and $P(x)$ is evenly divisible by $C(x)$)
    - ($P(x) + E(x)$) is exactly divisible by $C(x)$, error will not be detected
  - If result = 1, errors.

CRC – Example Encoding

$C(x) = x^3 + x^2 + 1 = 1101$ Generator

$M(x) = x^7 + x^4 + x^3 + x = 10011010$ Message

```
1101 ) 10011010000
       1101
       ----
        1001
        1101
        ----
         1000
         1101
         ----
          1011
          1101
          ----
           1100
           1101
           ----
             1000
             1101
             ----
              101
```

Message plus k zeros

k + 1 bit check sequence c, equivalent to a degree-k polynomial

Remainder

m mod c

Result:

Transmit message followed by remainder:

10011010101

CRC – Example Decoding – No Errors

$C(x) =$	$x^3 + x^2 + 1$	$= 1101$	**Generator**
$P(x) =$	$x^{10} + x^7 + x^6 + x^4 + x^2 + 1$	$= 10011010101$	**Received Message**

```
1101 | 10011010101   <--  Received message, no errors
       1101
       ----
        1001
        1101
        ----
         1000
         1101
         ----
          1011
          1101
          ----
           1100
           1101
           ----
             1101
             1101
             ----
                0
```

k + 1 bit check sequence c, equivalent to a degree-k polynomial

Remainder m mod c

Result:

CRC test is passed

CRC – Example Decoding – with Errors

C(x) = $x^3 + x^2 + 1$ **= 1101** **Generator**

P(x) = $x^{10} + x^7 + x^5 + x^4 + x^2 + 1$ **= 10010110101** **Received Message**

```
1101 | 10010110101   <- Received message
       1101          <- Two bit errors
       ----
        1000
        1101
        ----
         1011
         1101
         ----
          1101
          1101
          ----
           0101
```

k + 1 bit check sequence c, equivalent to a degree-k polynomial

Remainder

m mod c

Result:

CRC test failed

CRC Error Detection

- Properties
  - Characterize error as $E(x)$
  - Error detected unless $C(x)$ divides $E(x)$
    - (*i.e.*, $E(x)$ is a multiple of $C(x)$)

CRC Error Detection

- What errors can we detect?
  - All single-bit errors, if x^k and x^0 have non-zero coefficients
  - All double-bit errors, if C(x) has at least three terms
  - All odd bit errors, if C(x) contains the factor (x + 1)
  - Any bursts of length < k, if C(x) includes a constant term
  - Most bursts of length ≥ k

Common Polynomials for C(x)

CRC	C(x)
CRC-8	$x^8 + x^2 + x^1 + 1$
CRC-10	$x^{10} + x^9 + x^5 + x^4 + x^1 + 1$
CRC-12	$x^{12} + x^{11} + x^3 + x^2 + x^1 + 1$
CRC-16	$x^{16} + x^{15} + x^2 + 1$
CRC-CCITT	$x^{16} + x^{12} + x^5 + 1$
CRC-32	$x^{32} + x^{26} + x^{23} + x^{22} + x^{16} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^5 + x^4 + x^2 + x^1 + 1$

Error Detection vs. Error Correction

- Detection
  - Pro: Overhead only on messages with errors
  - Con: Cost in bandwidth and latency for retransmissions
- Correction
  - Pro: Quick recovery
  - Con: Overhead on all messages
- What should we use?
  - Correction if retransmission is too expensive
  - Correction if probability of errors is high
  - Detection when retransmission is easy and probability of errors is low

Detection vs. Correction: Cost, Overhead, Time

Dimension	Detection + ARQ	Correction (FEC)
Overhead per message	Small — a few check bits (e.g., CRC-32 = 32 bits)	Larger — enough redundancy to fix errors, not just flag them
Cost when channel is clean	Near zero beyond the check bits	Same overhead paid on every message, error or not
Cost when an error occurs	Extra round trip + retransmission bandwidth	None — fixed immediately, no retransmission
Latency / time	Unpredictable — grows with error rate and RTT	Predictable — one-pass decode, no round trip
Computation	Cheap — checksum/CRC calculation	Expensive — complex decoders (e.g., Viterbi, LDPC)
Best fit	Low error rate, cheap/fast retransmission (wired LAN)	High error rate or costly retransmission (satellite, deep space, broadcast, real-time media)