Problem 1 [20 points]

Consider the $2^{nd}$ column of the Fourier matrix, which is $[e^{j0} \ e^{j\theta} \ e^{j2\theta} \ ... \ e^{j(N-1)\theta}]^T$.

(a) Prove that the $3^{rd}$ column is orthogonal to the $2^{nd}$ column.

(b) Prove that any column is orthogonal to the $2^{nd}$ column.

(c) Prove that any two columns are orthogonal.

Problem 2 [25 points]

Consider a $N$ dimensional vector $\bar{v}$ expressed in the identity basis.

(a) Express the vector $\bar{v}$ in an orthonormal basis $F$, where $F$ is a $N \times N$ matrix.  

(Hint: See class notes on how we express a signal in different basis.)

(b) Let’s call the above vector $\bar{w}$. Now create a matrix $B$ such that $B\bar{w}$ scales the $i^{th}$ element of $\bar{w}$ by a scalar $b_i$. What should be the matrix $B$?

(c) Let’s denote the vector $B\bar{w}$ as vector $\bar{z}$. Now convert vector $\bar{z}$ back into the original identity basis.

(d) Now write all the above operations on vector $\bar{v}$ in one equation in terms of $F$ and $B$.

(e) Write the Eigen-decomposition equation of a matrix $A$, where $S$ contains the eigenvectors of $A$ and $\Lambda$ is the diagonal matrix containing the eigenvalues.

(f) Given the above exercise you have done, explain in plain English what Eigen-decomposition does to a vector (in other words, what happens when matrix $A$ is multiplied to vector $x$)?

Problem 3 [5 points]

In class, we discussed the analogy of expressing a job-interview candidate, Albert, in two different bases; one was the $\langle \text{math, programming, presentation} \rangle$ and the other was $\langle \text{machine learning, logic design, project report} \rangle$.

(a) Can you come up with another analogy from the real world where the same “thing” is expressed in 2 different “bases”. You must write that one “thing” and the 2 “bases”.

(b) In class, we called our analogy the Space-X transform. Please name your own transform for the analogy you came up with above.
Problem 4  [20 points]

(a) You are sampling a signal every 0.25 millisecond. What is the maximum frequency you would be able to see in FFT?

(b) Suppose you take \( N = 1000 \) for your FFT. At what frequency resolution would you be able to analyze the signal you are sampling? A frequency resolution of \( R \) Hz means you express the signal at frequencies \([0, R, 2R, ...] \) Hz.

(c) True/False : The FFT of any signal is symmetric around frequency zero. Explain your answer in 1 sentence.

(d) Say \( X_f \) is the DFT of a signal \( x_n \). Now, consider \( Y_f = X_f.e^{j\phi} \), where \( \phi \) is a constant angle. Is the IDFT\((Y_f)\) a shifted version of the signal \( x_n \)? Briefly argue in favor or against.

Problem 5  [20 points]

(a) Consider a signal \( x[n] = \cos(2\pi f_1 n t_s) + 2\sin(2\pi f_1 n t_s) + \sin(4\pi f_1 n t_s) \). Draw the magnitude and phase plots of \( X_f \), which is the DFT of \( x[n] \). Assume that \( f_1 \) is the fundamental frequency in which you are sampling the signal.

(b) Prove that DFT is linear, i.e., \( \text{DFT}(a_1 x[n] + a_2 y[n]) = a_1 X_f + a_2 Y_f \), where \( X_f \) and \( Y_f \) are the DFTs of \( x[n] \) and \( y[n] \), respectively.

Problem 6  [10 points]

Use your phone to record your own voice, and say “My name is [Your Name].” Use Python to import the saved audio file, and compute its FFT. Submit the plot of the magnitude of the FFT.

Hint: You can use `scipy.io.wavfile.read` to read the audio file and get the sampling rate. If your data has two channels, you can extract 1 with `data = data[:, 0]`. You can then compute the FFT with `scipy.fft`.

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