Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

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Programming Languages & Compilers

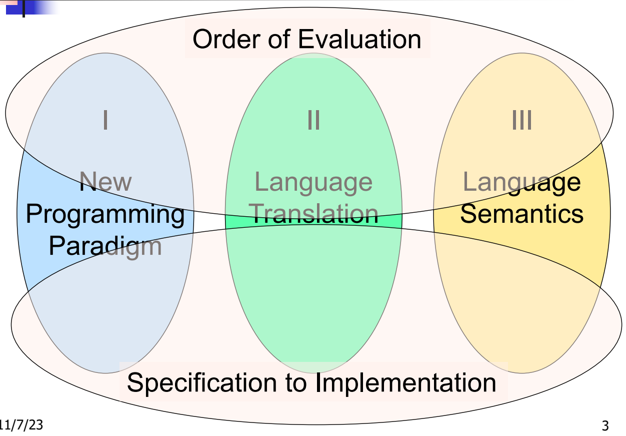
Three Main Topics of the Course

I
New Programming Paradigm

II
Language Translation

III
Language Semantics

Programming Languages & Compilers



Programming Languages & Compilers

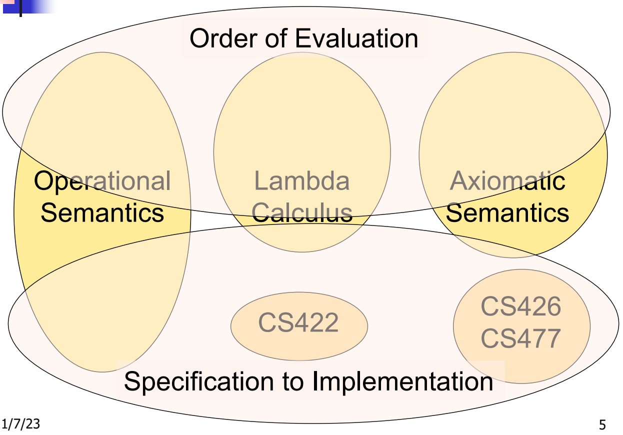
III : Language Semantics

Operational Semantics

Lambda Calculus

Axiomatic Semantics

Programming Languages & Compilers



Semantics

- Expresses the meaning of syntax
- Static semantics
  - Meaning based only on the form of the expression without executing it
  - Usually restricted to type checking / type inference

Dynamic semantics

- Method of describing meaning of executing a program
- Several different types:
  - Operational Semantics
  - Axiomatic Semantics
  - Denotational Semantics

Dynamic Semantics

- Different languages better suited to different types of semantics
- Different types of semantics serve different purposes

Operational Semantics

- Start with a simple notion of machine
- Describe how to execute (implement) programs of language on virtual machine, by describing how to execute each program statement (ie, following the *structure* of the program)
- Meaning of program is how its execution changes the state of the machine
- Useful as basis for implementations

Axiomatic Semantics

- Also called Floyd-Hoare Logic
- Based on formal logic (first order predicate calculus)
- Axiomatic Semantics is a logical system built from *axioms* and *inference rules*
- Mainly suited to simple imperative programming languages

Axiomatic Semantics

- Used to formally prove a property (*post-condition*) of the *state* (the values of the program variables) after the execution of program, assuming another property (*pre-condition*) of the state before execution
- Written :

  {Precondition} Program {Postcondition}
- Source of idea of *loop invariant*

Denotational Semantics

- Construct a function $\mathcal{M}$ assigning a mathematical meaning to each program construct
- Lambda calculus often used as the range of the meaning function
- Meaning function is compositional: meaning of construct built from meaning of parts
- Useful for proving properties of programs

Natural Semantics

- Aka Structural Operational Semantics, aka "Big Step Semantics"
- Provide value for a program by rules and derivations, similar to type derivations
- Rule conclusions look like

$$(C, m) \Downarrow m'$$

or

$$(E, m) \Downarrow v$$

Simple Imperative Programming Language

- $I \in$ *Identifiers*
- $N \in$ *Numerals*
- B ::= true | false | B & B | B or B | not B | $E < E$ | $E = E$
- $E ::= N \mid I \mid E + E \mid E * E \mid E - E \mid - E \mid (E)$
- C::= skip | C;C | I := E | if B then C else C fi | while B do C od

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Natural Semantics of Atomic Expressions

- Identifiers: $(I,m) \Downarrow m(I)$
- Numerals are values: $(N,m) \Downarrow N$
- Booleans: $(\text{true},m) \Downarrow \text{true}$
  $(\text{false} ,m) \Downarrow \text{false}$

Booleans:

$$\frac{(B, m) \Downarrow \text{false}}{(B \,\&\, B', m) \Downarrow \text{false}} \qquad \frac{(B, m) \Downarrow \text{true} \quad (B', m) \Downarrow b}{(B \,\&\, B', m) \Downarrow b}$$

$$\frac{(B, m) \Downarrow \text{true}}{(B \text{ or } B', m) \Downarrow \text{true}} \qquad \frac{(B, m) \Downarrow \text{false} \quad (B', m) \Downarrow b}{(B \text{ or } B', m) \Downarrow b}$$

$$\frac{(B, m) \Downarrow \text{true}}{(\text{not } B, m) \Downarrow \text{false}} \qquad \frac{(B, m) \Downarrow \text{false}}{(\text{not } B, m) \Downarrow \text{true}}$$

Relations

$$\frac{(E, m) \Downarrow U \quad (E', m) \Downarrow V \quad U \sim V = b}{(E \sim E', m) \Downarrow b}$$

- By $U \sim V = b$, we mean does (the meaning of) the relation $\sim$ hold on the meaning of U and V
- May be specified by a mathematical expression/equation or rules matching U and V

Arithmetic Expressions

$$\frac{(E, m) \Downarrow U \quad (E', m) \Downarrow V \quad U \text{ op } V = N}{(E \text{ op } E', m) \Downarrow N}$$

where N is the specified value for U *op* V

Commands

Skip: $(\text{skip}, m) \Downarrow m$

Assignment:
$$\frac{(E,m) \Downarrow V}{(I{:=}E,m) \Downarrow m[I \text{ <-- } V] \ {\color{red}(=\{\text{I -> V}\}+\text{m})}}$$

Sequencing:
$$\frac{(C,m) \Downarrow m' \quad (C',m') \Downarrow m''}{(C;C',\ m) \Downarrow m''}$$

If Then Else Command

$$\frac{(B,m) \Downarrow \text{true} \quad (C,m) \Downarrow m'}{(\text{if } B \text{ then } C \text{ else } C' \text{ fi, } m) \Downarrow m'}$$

$$\frac{(B,m) \Downarrow \text{false} \quad (C',m) \Downarrow m'}{(\text{if } B \text{ then } C \text{ else } C' \text{ fi, } m) \Downarrow m'}$$

While Command

$$\frac{(B,m) \Downarrow \text{false}}{(\text{while } B \text{ do } C \text{ od, } m) \Downarrow m}$$

$$\frac{(B,m)\Downarrow\text{true} \ (C,m)\Downarrow m' \quad (\text{while } B \text{ do } C \text{ od, } m')\Downarrow m''}{(\text{while } B \text{ do C od, } m) \Downarrow m''}$$

Example: If Then Else Rule

$$\frac{\quad}{\begin{array}{c}(\text{if x > 5 then y:= 2 + 3 else y:=3 + 4 fi,}\\ \{\text{x -> 7}\}) \Downarrow ?\end{array}}$$

Example: If Then Else Rule

$$\frac{\dfrac{\quad}{(\text{x > 5, }\{\text{x -> 7}\})\Downarrow ?}}{\begin{array}{c}(\text{if x > 5 then y:= 2 + 3 else y:=3 + 4 fi,}\\ \{\text{x -> 7}\}) \Downarrow \ ?\end{array}}$$

Example: Arith Relation

$$\frac{\dfrac{\begin{array}{c} ? > ? = ? \\ (\text{x,}\{\text{x->7}\})\Downarrow ? \quad (\text{5,}\{\text{x->7}\})\Downarrow ?\end{array}}{(\text{x > 5, }\{\text{x -> 7}\})\Downarrow ?}}{\begin{array}{c}(\text{if x > 5 then y:= 2 + 3 else y:=3 + 4 fi,}\\ \{\text{x -> 7}\}) \Downarrow ?\end{array}}$$

Example: Identifier(s)

$$\dfrac{\dfrac{\begin{array}{c}7 > 5 = \text{true}\\ (x,\{x\text{->}7\})\Downarrow 7 \quad (5,\{x\text{->}7\})\Downarrow 5\end{array}}{(x > 5, \{x \text{->} 7\})\Downarrow ?}}{\begin{array}{c}(\text{if } x > 5 \text{ then } y{:}= 2 + 3 \text{ else } y{:}=3 + 4 \text{ fi},\\ \{x \text{->} 7\}) \Downarrow ?\end{array}}$$

Example: Arith Relation

$$\dfrac{\dfrac{\begin{array}{c}7 > 5 = \text{true}\\ (x,\{x\text{->}7\})\Downarrow 7 \quad (5,\{x\text{->}7\})\Downarrow 5\end{array}}{(x > 5, \{x \text{->} 7\})\Downarrow \text{true}}}{\begin{array}{c}(\text{if } x > 5 \text{ then } y{:}= 2 + 3 \text{ else } y{:}=3 + 4 \text{ fi},\\ \{x \text{->} 7\}) \Downarrow ?\end{array}}$$

Example: If Then Else Rule

$$\dfrac{\dfrac{\begin{array}{c}7 > 5 = \text{true}\\ (x,\{x\text{->}7\})\Downarrow 7 \quad (5,\{x\text{->}7\})\Downarrow 5\end{array}}{(x > 5, \{x \text{->} 7\})\Downarrow \text{true}} \quad \dfrac{}{\begin{array}{c}(y{:}= 2 + 3, \{x\text{->} 7\}\\ \Downarrow ? \qquad .\end{array}}}{\begin{array}{c}(\text{if } x > 5 \text{ then } y{:}= 2 + 3 \text{ else } y{:}=3 + 4 \text{ fi},\\ \{x \text{->} 7\}) \Downarrow ?\end{array}}$$

Example: Assignment

$$\dfrac{\dfrac{\begin{array}{c}7 > 5 = \text{true}\\ (x,\{x\text{->}7\})\Downarrow 7 \quad (5,\{x\text{->}7\})\Downarrow 5\end{array}}{(x > 5, \{x \text{->} 7\})\Downarrow \text{true}} \quad \dfrac{\dfrac{}{(2+3, \{x\text{->}7\})\Downarrow ?}}{\begin{array}{c}(y{:}= 2 + 3, \{x\text{->} 7\}\\ \Downarrow ?\end{array}}}{\begin{array}{c}(\text{if } x > 5 \text{ then } y{:}= 2 + 3 \text{ else } y{:}=3 + 4 \text{ fi},\\ \{x \text{->} 7\}) \Downarrow ?\end{array}}$$

Example: Arith Op

$$\dfrac{\dfrac{\begin{array}{c}7 > 5 = \text{true}\\ (x,\{x\text{->}7\})\Downarrow 7 \quad (5,\{x\text{->}7\})\Downarrow 5\end{array}}{(x > 5, \{x \text{->} 7\})\Downarrow \text{true}} \quad \dfrac{\dfrac{\begin{array}{c}? + ? = ?\\ (2,\{x\text{->}7\})\Downarrow ? \quad (3,\{x\text{->}7\}) \Downarrow ?\end{array}}{(2+3, \{x\text{->}7\})\Downarrow ?}}{\begin{array}{c}(y{:}= 2 + 3, \{x\text{->} 7\}\\ \Downarrow ? \qquad .\end{array}}}{\begin{array}{c}(\text{if } x > 5 \text{ then } y{:}= 2 + 3 \text{ else } y{:}=3 + 4 \text{ fi},\\ \{x \text{->} 7\}) \Downarrow ?\end{array}}$$

Example: Numerals

$$\dfrac{\dfrac{\begin{array}{c}7 > 5 = \text{true}\\ (x,\{x\text{->}7\})\Downarrow 7 \quad (5,\{x\text{->}7\})\Downarrow 5\end{array}}{(x > 5, \{x \text{->} 7\})\Downarrow \text{true}} \quad \dfrac{\dfrac{\begin{array}{c}2 + 3 = 5\\ (2,\{x\text{->}7\})\Downarrow 2 \quad (3,\{x\text{->}7\}) \Downarrow 3\end{array}}{(2+3, \{x\text{->}7\})\Downarrow ?}}{\begin{array}{c}(y{:}= 2 + 3, \{x\text{->} 7\}\\ \Downarrow ?\end{array}}}{\begin{array}{c}(\text{if } x > 5 \text{ then } y{:}= 2 + 3 \text{ else } y{:}=3 + 4 \text{ fi},\\ \{x \text{->} 7\}) \Downarrow ?\end{array}}$$

Example: Arith Op

2 + 3 = 5
(2,{x->7})⇓2 (3,{x->7}) ⇓3
7 > 5 = true (2+3, {x->7})⇓5
(x,{x->7})⇓7 (5,{x->7})⇓5 (y:= 2 + 3, {x-> 7}
(x > 5, {x -> 7})⇓true ⇓?
(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi,
{x -> 7}) ⇓ ?

Example: Assignment

2 + 3 = 5
(2,{x->7})⇓2 (3,{x->7}) ⇓3
7 > 5 = true (2+3, {x->7})⇓5
(x,{x->7})⇓7 (5,{x->7})⇓5 (y:= 2 + 3, {x-> 7}
(x > 5, {x -> 7})⇓true ⇓ {x->7, y->5}
(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi,
{x -> 7}) ⇓?

Example: If Then Else Rule

2 + 3 = 5
(2,{x->7})⇓2 (3,{x->7}) ⇓3
7 > 5 = true (2+3, {x->7})⇓5
(x,{x->7})⇓7 (5,{x->7})⇓5 (y:= 2 + 3, {x-> 7}
(x > 5, {x -> 7})⇓true ⇓ {x->7, y->5}
(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi,
{x -> 7}) ⇓ {x->7, y->5}

Comment

- Simple Imperative Programming Language introduces variables *implicitly* through assignment
- The let-in command introduces scoped variables *explictly*
- Clash of constructs apparent in awkward semantics

Interpretation Versus Compilation

- A **compiler** from language L1 to language L2 is a program that takes an L1 program and for each piece of code in L1 generates a piece of code in L2 of same meaning
- An **interpreter** of L1 in L2 is an L2 program that executes the meaning of a given L1 program
- Compiler would examine the body of a loop once; an interpreter would examine it every time the loop was executed

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Interpreter

- An *Interpreter* represents the operational semantics of a language L1 (source language) in the language of implementation L2 (target language)
- Built incrementally
  - Start with literals
  - Variables
  - Primitive operations
  - Evaluation of expressions
  - Evaluation of commands/declarations

Interpreter

- Takes abstract syntax trees as input
  - In simple cases could be just strings
- One procedure for each syntactic category (nonterminal)
  - eg one for expressions, another for commands
- If Natural semantics used, tells how to compute final value from code
- If Transition semantics used, tells how to compute next “state”
  - To get final value, put in a loop

Natural Semantics Example

- compute_exp (Var(v), m) = look_up v m
- compute_exp (Int(n), _) = Num (n)
- …
- compute_com(IfExp(b,c1,c2),m) =

  if compute_exp (b,m) = Bool(true)

  then compute_com (c1,m)

  else compute_com (c2,m)

Natural Semantics Example

- compute_com(While(b,c), m) =

  if compute_exp (b,m) = Bool(false)

  then m

  else compute_com

  (While(b,c), compute_com(c,m))

- May fail to terminate - exceed stack limits
- Returns no useful information then

Transition Semantics

- Form of operational semantics
- Describes how each program construct transforms machine state by *transitions*
- Rules look like

  (C, *m*) --> (C’ , *m’*) or (C,*m*) --> *m’*
- *C, C’* is code remaining to be executed
- m, m’ represent the state/store/memory/environment
  - Partial mapping from identifiers to values
  - Sometimes *m* (or *C*) not needed
- Indicates exactly one step of computation

Expressions and Values

- *C, C’* used for commands; *E, E’ for* expressions; *U,V* for values
- Special class of expressions designated as *values*
  - Eg 2, 3 are values, but 2+3 is only an expression
- Memory only holds values
  - Other possibilities exist

Evaluation Semantics

- Transitions successfully stops when E/*C* is a value/memory
- Evaluation fails if no transition possible, but not at value/memory
- Value/memory is the final *meaning* of original expression/command (in the given state)
- Coarse semantics: final value / memory
- More fine grained: whole transition sequence

Simple Imperative Programming Language

- $I \in$ *Identifiers*
- $N \in$ *Numerals*
- *B* ::= true | false | *B* & *B* | *B* or *B* | not *B* | *E* < *E* | *E* = *E*
- *E::= N | I | E + E | E * E | E - E | - E*
- C::= skip | *C*;*C* | *I* ::= *E*
  | if *B* then *C* else *C* fi | while *B* do *C* od

Transitions for Expressions

- Numerals are values
- Boolean values = {true, false}
- Identifiers: (*I*,*m*) --> (*m*(*I*), *m*)

Boolean Operations:

- Operators: (short-circuit)

(false & *B*, *m*) --> (false,*m*)
(true & *B*, *m*) --> (*B*,*m*)

$$\frac{(B,\ m) \text{ --> } (B'',\ m)}{(B \ \&\ B',\ m) \text{ --> } (B'' \&\ B',\ m)}$$

(true or *B*, *m*) --> (true,*m*)
(false or *B*, *m*) --> (*B*,*m*)

$$\frac{(B,\ m) \text{ --> } (B'',\ m)}{(B \text{ or } B',\ m) \text{ --> } (B'' \text{ or } B',m)}$$

(not true, m) --> (false,*m*)
(not false, m) --> (true,*m*)

$$\frac{(B,\ m) \text{ --> } (B',\ m)}{(\text{not } B,\ m) \text{ --> } (\text{not } B',\ m)}$$

Relations

$$\frac{(E,\ m) \text{ --> } (E'',m)}{(E \sim E',\ m) \text{ --> } (E''{\sim}E',m)}$$

$$\frac{(E,\ m) \text{ --> } (E',m)}{(V \sim E,\ m) \text{ --> } (V{\sim}E',m)}$$

(*U* ~ *V*, *m*) --> (true,*m*) or (false,*m*)
depending on whether *U* ~ *V* holds or not

Arithmetic Expressions

$$\frac{(E,\ m) \text{ --> } (E'',m)}{(E \ op \ E',\ m) \text{ --> } (E'' \ op \ E',m)}$$

$$\frac{(E,\ m) \text{ --> } (E',m)}{(V \ op \ E,\ m) \text{ --> } (V \ op \ E',m)}$$

(*U op V*, *m*) --> (*N*,*m*) where *N* is the specified value for *U op V*

Commands - in English

- skip means done evaluating
- When evaluating an assignment, evaluate the expression first
- If the expression being assigned is already a value, update the memory with the new value for the identifier
- When evaluating a sequence, work on the first command in the sequence first
- If the first command evaluates to a new memory (ie completes), evaluate remainder with new memory

Commands

$$(\text{skip}, m) \text{ --> } m$$

$$\frac{(E,m) \text{ --> } (E',m)}{(I::=E,m) \text{ --> } (I::=E',m)}$$

$$(I::=V,m) \text{ --> } m[I <\text{-- } V]$$

$$\frac{(C,m) \text{ --> } (C'',m')}{(C;C',\ m) \text{ --> } (C'';C',m')} \qquad \frac{(C,m) \text{ --> } m'}{(C;C',\ m) \text{ --> } (C',m')}$$

If Then Else Command - in English

- If the boolean guard in an if_then_else is true, then evaluate the first branch
- If it is false, evaluate the second branch
- If the boolean guard is not a value, then start by evaluating it first.

If Then Else Command

$$(\text{if true then } C \text{ else } C' \text{ fi},\ m) \text{ --> } (C,\ m)$$

$$(\text{if false then } C \text{ else } C' \text{ fi},\ m) \text{ --> } (C',\ m)$$

$$\frac{(B,m) \text{ --> } (B',m)}{\begin{array}{c}(\text{if } B \text{ then } C \text{ else } C' \text{ fi},\ m) \\ \text{--> } (\text{if } B' \text{ then } C \text{ else } C' \text{ fi},\ m)\end{array}}$$

What should while transition to?

$$\frac{}{(\text{while B do C od, m}) \rightarrow ?}$$

Wrong! BAD

$$\frac{(\text{B, m}) \rightarrow (\text{B}',\text{ m})}{(\text{while B do C od, m}) \text{ -}\rightarrow (\text{while B}' \text{ do C od, m})}$$

While Command

$$(\text{while } B \text{ do } C \text{ od},\ m) \text{ --> }$$
$$(\text{if } B \text{ then } C;\ \text{while } B \text{ do } C \text{ od else skip fi, m})$$

In English: Expand a While into a test of the boolean guard, with the true case being to do the body and then try the while loop again, and the false case being to stop.

Example Evaluation

- First step:

$$\frac{}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> ?}}$$

Example Evaluation

- First step:

$$\frac{\text{(x > 5, \{x -> 7\}) --> ?}}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> ?}}$$

Example Evaluation

- First step:

$$\frac{\dfrac{\text{(x,\{x -> 7\}) --> (7, \{x -> 7\})}}{\text{(x > 5, \{x -> 7\}) --> ?}}}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> ?}}$$

Example Evaluation

- First step:

$$\frac{\dfrac{\text{(x,\{x -> 7\}) --> (7, \{x -> 7\})}}{\text{(x > 5, \{x -> 7\}) --> (7 > 5, \{x -> 7\})}}}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> ?}}$$

Example Evaluation

- First step:

$$\frac{\dfrac{\text{(x,\{x -> 7\}) --> (7, \{x -> 7\})}}{\text{(x > 5, \{x -> 7\}) --> (7 > 5, \{x -> 7\})}}}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\})}}$$

$$\text{--> (if 7 > 5 then y:=2 + 3 else y:=3 + 4 fi, \{x -> 7\})}$$

Example Evaluation

- Second Step:

$$\frac{\text{(7 > 5, \{x -> 7\}) --> (true, \{x -> 7\})}}{\text{(if 7 > 5 then y:=2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> (if true then y:=2 + 3 else y:=3 + 4 fi, \{x -> 7\})}}$$

- Third Step:

(if true then y:=2 + 3 else y:=3 + 4 fi, {x -> 7})
-->(y:=2+3, {x->7})

Example Evaluation

- Fourth Step:

$$\frac{(2+3,\ \{x\text{-> } 7\})\ \text{-->}\ (5,\ \{x\text{ -> } 7\})}{(y{:=}2+3,\ \{x\text{->}7\})\ \text{-->}\ (y{:=}5,\ \{x\text{->}7\})}$$

- Fifth Step:

(y:=5, {x->7}) --> {y -> 5, x -> 7}

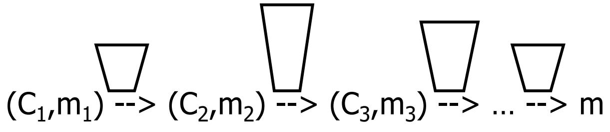
Example Evaluation

- Bottom Line:

(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi,
{x -> 7})
--> (if 7 > 5 then y:=2 + 3 else y:=3 + 4 fi,
{x -> 7})
-->(if true then y:=2 + 3 else y:=3 + 4 fi,
{x -> 7})
-->(y:=2+3, {x->7})
--> (y:=5, {x->7}) --> {y -> 5, x -> 7}

Transition Semantics Evaluation

- A sequence of steps with trees of justification for each step

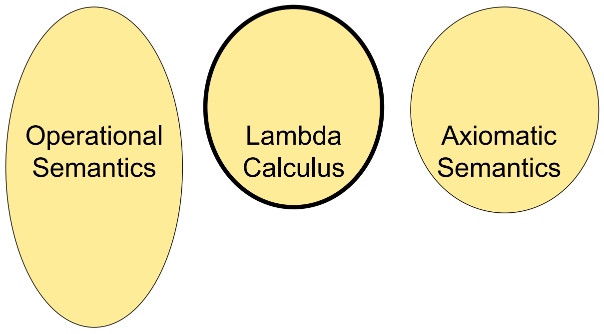


(C_1,m_1) --> (C_2,m_2) --> (C_3,m_3) --> … --> m

- Let -->* be the transitive closure of -->
- Ie, the smallest transitive relation containing -->

Programming Languages & Compilers

III : Language Semantics



Operational Semantics

Lambda Calculus

Axiomatic Semantics

Lambda Calculus - Motivation

- Aim is to capture the essence of functions, function applications, and evaluation
- λ–calculus is a theory of computation
- “The Lambda Calculus: Its Syntax and Semantics”. H. P. Barendregt. North Holland, 1984

Lambda Calculus - Motivation

- All *sequential programs* may be viewed as functions from input (initial state and input values) to output (resulting state and output values).
- λ-calculus is a mathematical formalism of functions and functional computations
- Two flavors: typed and untyped

Untyped λ-Calculus

- Only three kinds of expressions:
  - Variables: x, y, z, w, …
  - Abstraction: λ x. e

    (Function creation, think fun x -> e)
  - Application: e_1 e_2
  - Parenthesized expression: (e)

Untyped λ-Calculus Grammar

- Formal BNF Grammar:
  - <expression> ::= <variable>

    | <abstraction>

    | <application>

    | (<expression>)
  - <abstraction>

    ::= λ<variable>.<expression>
  - <application>

    ::= <expression> <expression>

Untyped λ-Calculus Terminology

- Occurrence: a location of a subterm in a term
- Variable binding: λ x. e is a binding of x in e
- Bound occurrence: all occurrences of x in λ x. e
- Free occurrence: one that is not bound
- Scope of binding: in λ x. e, all occurrences in e not in a subterm of the form λ x. e′ (same x)
- Free variables: all variables having free occurrences in a term

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Example

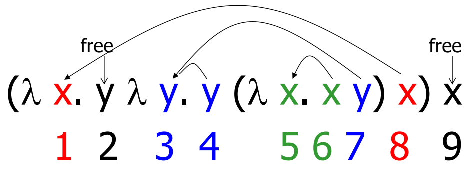
- Label occurrences and scope:

(λ x. y λ y. y (λ x. x y) x) x

1 2 3 4 5 6 7 8 9

Example

- Label occurrences and scope:



(λ x. y λ y. y (λ x. x y) x) x

1 2 3 4 5 6 7 8 9

Untyped λ-Calculus

- How do you compute with the λ-calculus?
- Roughly speaking, by substitution:
- (λ x. e_1) e_2 ⇒* e_1 [e_2 / x]
- * Modulo all kinds of subtleties to avoid free variable capture

Transition Semantics for λ-Calculus

$$\frac{E \text{ –> } E''}{E\ E' \text{ --> } E''\ E'}$$

- Application (version 1 - Lazy Evaluation)

$$(\lambda\ x\ .\ E)\ E' \text{ --> } E[E'/x]$$

- Application (version 2 - Eager Evaluation)

$$\frac{E' \text{ --> } E''}{(\lambda\ x\ .\ E)\ E' \text{ --> } (\lambda\ x\ .\ E)\ E''}$$

$$\frac{}{(\lambda\ x\ .\ E)\ V \text{--> } E[V/x]}$$

V - variable or abstraction (value)

How Powerful is the Untyped λ-Calculus?

- The untyped λ-calculus is Turing Complete
  - Can express any sequential computation
- Problems:
  - How to express basic data: booleans, integers, etc?
  - How to express recursion?
  - Constants, if_then_else, etc, are conveniences; can be added as syntactic sugar

Typed vs Untyped λ-Calculus

- The *pure* λ-calculus has no notion of type: (f f) is a legal expression
- Types restrict which applications are valid
- Types are not syntactic sugar! They disallow some terms
- Simply typed λ-calculus is less powerful than the untyped λ-Calculus: NOT Turing Complete (no recursion)

α Conversion

1. α-conversion:
   2. λ x. exp --α--> λ y. (exp [y/x])
3. Provided that
   1. y is not free in exp
   2. No free occurrence of x in exp becomes bound in exp when replaced by y

λ x. x (λ y. x y) - × -> λ y. y(λ y.y y)

α Conversion Non-Examples

1. Error: y is not free in term second

$$\lambda\ x.\ x\ y\ \cancel{\text{--}\alpha\text{--}}\text{>}\ \lambda\ y.\ y\ y$$

2. Error: free occurrence of x becomes bound in wrong way when replaced by y

$$\lambda\ x.\ \underbrace{\lambda\ y.\ x\ y}_{\text{exp}}\ \cancel{\text{--}\alpha\text{--}}\text{>}\ \lambda\ y.\ \underbrace{\lambda\ y.\ y\ y}_{\text{exp[y/x]}}$$

But λ x. (λ y. y) x --α--> λ y. (λ y. y) y

And λ y. (λ y. y) y --α--> λ x. (λ y. y) x

Congruence

- Let ~ be a relation on lambda terms. ~ is a congruence if
- it is an equivalence relation
- If $e_1 \sim e_2$ then
  - $(e\ e_1) \sim (e\ e_2)$ and $(e_1 e) \sim (e_2\ e)$
  - $\lambda\ x.\ e_1 \sim \lambda\ x.\ e_2$

α Equivalence

- α equivalence is the smallest congruence containing α conversion
- One usually treats α-equivalent terms as equal - i.e. use α equivalence classes of terms

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Example

Show: $\lambda x. (\lambda y. y\ x)\ x \sim\alpha\sim \lambda y. (\lambda x. x\ y)\ y$

- $\lambda x. (\lambda y. y\ x)\ x \text{ --}\alpha\text{--> } \lambda z. (\lambda y. y\ z)\ z$ so
  $\lambda x. (\lambda y. y\ x)\ x \sim\alpha\sim \lambda z. (\lambda y. y\ z)\ z$
- $(\lambda y. y\ z) \text{ --}\alpha\text{--> } (\lambda x. x\ z)$ so
  $(\lambda y. y\ z) \sim\alpha\sim (\lambda x. x\ z)$ so
  $(\lambda y. y\ z)\ z \sim\alpha\sim (\lambda x. x\ z)\ z$ so
  $\lambda z. (\lambda y. y\ z)\ z \sim\alpha\sim \lambda z. (\lambda x. x\ z)\ z$
- $\lambda z. (\lambda x. x\ z)\ z \text{ --}\alpha\text{--> } \lambda y. (\lambda x. x\ y)\ y$ so
  $\lambda z. (\lambda x. x\ z)\ z \sim\alpha\sim \lambda y. (\lambda x. x\ y)\ y$
- $\lambda x. (\lambda y. y\ x)\ x \sim\alpha\sim \lambda y. (\lambda x. x\ y)\ y$