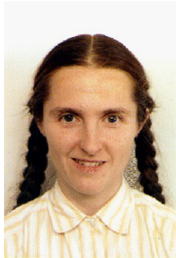
Programming Languages and Compilers (CS 421)

Elsa L Gunter

2112 SC, UIUC



https://courses.engr.illinois.edu/cs421/fa2017/CS421D

Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Evaluating declarations

- Evaluation uses an environment ρ
- To evaluate a (simple) declaration let x = e
  - Evaluate expression e in ρ to value v
  - Update ρ with $x \to v$: $\{x \to v\} + \rho$

Evaluating declarations

- Evaluation uses an environment ρ
- To evaluate a (simple) declaration let $x = e$
  - Evaluate expression e in ρ to value v
  - Update ρ with x v: $\{x \to v\} + \rho$

- Update: $\rho_1 + \rho_2$ has all the bindings in ρ_1 and all those in ρ_2 that are not rebound in ρ_1

$\{x \to 2, y \to 3, a \to \text{"hi"}\} + \{y \to 100, b \to 6\}$

$= \{x \to 2, y \to 3, a \to \text{"hi"}, b \to 6\}$

Evaluating expressions in OCaml

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- A constant evaluates to itself, including primitive operators like + and =
- To evaluate a variable, look it up in ρ: $\rho(v)$
- To evaluate a tuple $(e_1,\ldots,e_n)$,
  - Evaluate each e_i to v_i, right to left for Ocaml
  - Then make value $(v_1,\ldots,v_n)$

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Evaluating expressions in OCaml

- To evaluate uses of +, - , etc, eval args (right to left for Ocaml), then do operation
- Function expression evaluates to its closure
- To evaluate a local dec: let x = e1 in e2
  - Eval e1 to v, then eval e2 using $\{x \to v\} + \rho$
- To evaluate a conditional expression: if b then e1 else e2
  - Evaluate b to a value v
  - If v is True, evaluate e1
  - If v is False, evaluate e2

Evaluation of Application with Closures

- Given application expression $f\ e$
- In Ocaml, evaluate e to value v
- In environment ρ, evaluate left term to closure, $c = <(x_1,\ldots,x_n) \to b, \rho'>$
  - $(x_1,\ldots,x_n)$ variables in (first) argument
  - v must have form $(v_1,\ldots,v_n)$
- Update the environment ρ' to

  $\rho'' = \{x_1 \to v_1,\ldots, x_n \to v_n\} + \rho'$
- Evaluate body b in environment ρ''

Recursive Functions

```
# let rec factorial n =
    if n = 0 then 1 else n * factorial (n - 1);;
  val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
# (* rec  is needed for recursive function
   declarations *)
```

Recursion Example

Compute n^2 recursively using:

$$n^2 = (2 * n - 1) + (n - 1)^2$$

```
# let rec nthsq n =          (* rec for recursion *)
    match n                  (* pattern matching for cases *)
    with 0 -> 0              (* base case *)
    | n -> (2 * n -1)        (* recursive case *)
         + nthsq (n -1);;    (* recursive call *)
val nthsq : int -> int = <fun>
# nthsq 3;;
- : int = 9
```

Structure of recursion similar to inductive proof

Recursion and Induction

```
# let rec nthsq n = match n with 0 -> 0
        | n -> (2 * n - 1) + nthsq (n - 1) ;;
```

- Base case is the last case; it stops the computation
- Recursive call must be to arguments that are somehow smaller - must progress to base case
- **if** or **match** must contain base case
- Failure of these may cause failure of termination

Lists

- List can take one of two forms:
  - Empty list, written [ ]
  - Non-empty list, written x :: xs
    - x is head element, xs is tail list, :: called "cons"
  - Syntactic sugar: [x] == x :: [ ]
  - [x1; x2; …; xn] == x1 :: x2 :: … :: xn :: [ ]

Lists

```
# let fib5 = [8;5;3;2;1;1];;
val fib5 : int list = [8; 5; 3; 2; 1; 1]
# let fib6 = 13 :: fib5;;
val fib6 : int list = [13; 8; 5; 3; 2; 1; 1]
# (8::5::3::2::1::1::[ ]) = fib5;;
- : bool = true
# fib5 @ fib6;;
- : int list = [8; 5; 3; 2; 1; 1; 13; 8; 5; 3; 2; 1;
  1]
```

Lists are Homogeneous

```
# let bad_list = [1; 3.2; 7];;
Characters 19-22:
  let bad_list = [1; 3.2; 7];;
                     ^^^
This expression has type float but is here
  used with type int
```

Question

- Which one of these lists is invalid?

1. [2; 3; 4; 6]
2. [2,3; 4,5; 6,7]
3. [(2.3,4); (3.2,5); (6,7.2)]
4. [[“hi”; “there”]; [“wahcha”]; [ ]; [“doin”]]

Answer

- Which one of these lists is invalid?

1. [2; 3; 4; 6]
2. [2,3; 4,5; 6,7]
3. [(2.3,4); (3.2,5); (6,7.2)]
4. [[“hi”; “there”]; [“wahcha”]; [ ]; [“doin”]]

- 3 is invalid because of last pair

Functions Over Lists

```
# let rec double_up list =
    match list
    with [ ] -> [ ]  (* pattern before ->,
                        expression after *)
       | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>
# let fib5_2 = double_up fib5;;
val fib5_2 : int list = [8; 8; 5; 5; 3; 3; 2; 2; 1;
  1; 1; 1]
```

Functions Over Lists

```
# let silly = double_up ["hi"; "there"];;
val silly : string list = ["hi"; "hi"; "there"; "there"]
# let rec poor_rev list =
  match list
  with [] -> []
    | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
# poor_rev silly;;
- : string list = ["there"; "there"; "hi"; "hi"]
```

Structural Recursion

- Functions on recursive datatypes (eg lists) tend to be recursive
- Recursion over recursive datatypes generally by structural recursion
  - Recursive calls made to components of structure of the same recursive type
  - Base cases of recursive types stop the recursion of the function

Question: Length of list

- Problem: write code for the length of the list
  - How to start?

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let rec length list =
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    match list with [] ->
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Question: Length of list

- Problem: write code for the length of the list
  - What result do we give when list is not empty?

```
let rec length list =
   match list with [] -> 0
    | (a :: bs) -> 1 + length bs
```

Structural Recursion : List Example

```
# let rec length list = match list
  with [ ] -> 0   (* Nil case *)
  | a :: bs -> 1 + length bs;;  (* Cons case *)
val length : 'a list -> int = <fun>
# length [5; 4; 3; 2];;
- : int = 4
```

- Nil case [ ] is base case
- Cons case recurses on component list bs

Same Length

- How can we efficiently answer if two lists have the same length?

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   | (x::xs) ->
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let rec same_length list1 list2 =
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      (match list2 with [] -> true
        | (y::ys) -> false)
   | (x::xs) ->
```

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let rec same_length list1 list2 =
   match list1 with [] ->
     (match list2 with [] -> true
       | (y::ys) -> false)
   | (x::xs) ->
     (match list2 with [] -> false
       | (y::ys) -> same_length xs ys)
```

Your turn: doubleList : int list -> int list

- Write a function that takes a list of int and returns a list of the same length, where each element has been multiplied by 2

let rec doubleList list =

Your turn: doubleList : int list -> int list

- Write a function that takes a list of int and returns a list of the same length, where each element has been multiplied by 2

```
let rec doubleList list =
  match list
    with [] ->[]
      |  x :: xs -> (2 * x) :: doubleList xs
```

Your turn: doubleList : int list -> int list

- Write a function that takes a list of int and returns a list of the same length, where each element has been multiplied by 2

```
let rec doubleList list =
  match list
    with [] ->[]
      | x :: xs -> (2 * x) :: doubleList xs
```

Folding Recursion

- Another common form "folds" an operation over the elements of the structure

```
# let rec multList list = match list
  with [ ] -> 1
  | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;4;6];;
- : int = 48
```

- Computes (2 * (4 * (6 * 1)))

Folding Recursion : Length Example

```
# let rec length list = match list
  with [ ] -> 0   (* Nil case *)
  | a :: bs -> 1 + length bs;;  (* Cons case *)
val length : 'a list -> int = <fun>
# length [5; 4; 3; 2];;
- : int = 4
```

- Nil case [ ] is base case, 0 is the base value
- Cons case recurses on component list bs
- What do multList and length have in common?

Forward Recursion

- In Structural Recursion, split input into components and (eventually) recurse
- Forward Recursion form of Structural Recursion
- In forward recursion, **first** call the function recursively on all recursive components, and then build final result from partial results
- Wait until whole structure has been traversed to start building answer

Forward Recursion: Examples

```
# let rec double_up list =
    match list
    with [ ] -> [ ]
      | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>

# let rec poor_rev list =
  match list
  with [] -> []
    | (x::xs) -> let r = poor_rev xs in r @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Forward Recursion: Examples

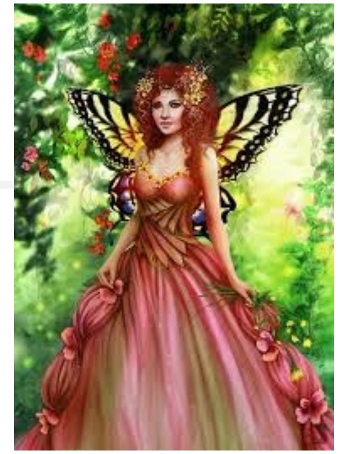
```
# let rec double_up list =
    match list
    with [ ] -> [ ]
       | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>
```

Base Case | Operator | Recursive Call

```
# let rec poor_rev list =
  match list
  with [] -> []
     | (x::xs) -> let r = poor_rev xs in r @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Base Case | Operator | Recursive Call

Recursing over lists



The Primitive Recursion Fairy

```
# let rec fold_right f list b =
  match list
  with [] -> b
  | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
   <fun>
# fold_right
    (fun s -> fun () -> print_string s)
    ["hi"; "there"]
    ();;
therehi- : unit = ()
```

Folding Recursion : Length Example

```
# let rec length list = match list
  with [ ] -> 0   (* Nil case *)
  | a :: bs -> 1 + length bs;;  (* Cons case *)
val length : 'a list -> int = <fun>
# let length list =
fold_right (fun a -> fun r -> 1 + r) list 0;;
val length : 'a list -> int = <fun>
# length [5; 4; 3; 2];;
- : int = 4
```

Forward Recursion: Examples

```
# let rec double_up list =
    match list
    with [ ] -> [ ]
       | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>
```

Base Case   Operator   Recursive Call

```
# let double_up =
    fold_right (fun x -> fun r -> x :: x :: r) list [ ]
```

Operator   Recursive result   Base Case

```
# double_up  ["a";"b"];;
- : string list = ["a"; "a"; "b"; "b"]
```

- let rec multList_fr list =
  match list

```
with [] -> 1
| (x::xs) -> let r = (multList_fr ns) in
                  (x * r)
```

Folding Recursion

- multList folds to the right
- Same as:

```
# let multList list =
    List.fold_right
    (fun x -> fun p -> x * p)
    list 1;;
val multList : int list -> int = <fun>
# multList [2;4;6];;
- : int = 48
```

Terminology

- Available: A function call that can be executed by the current expression
- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).
  - if $\boxed{\text{(h x)}}$ then $\boxed{\text{f x}}$ else $\boxed{\text{(x + g x)}}$
  - if $\boxed{\text{(h x)}}$ then (fun x -> f x) else $\boxed{\text{(g (x + x))}}$

Not available

Terminology

- Tail Position: A subexpression s of expressions e, which is available and such that if evaluated, will be taken as the value of e (last thing done in this expression)
  - if (x>3) then $\boxed{\text{x + 2}}$ else $\boxed{\text{x - 4}}$
  - let x = 5 in $\boxed{\text{x + 4}}$
- Tail Call: A function call that occurs in tail position
  - if (h x) then $\boxed{\text{f x}}$ else $\boxed{(\text{x} \pm \text{g x})}$

Tail Recursion

- A recursive program is tail recursive if all recursive calls are tail calls
- Tail recursive programs may be optimized to be implemented as loops, thus removing the function call overhead for the recursive calls
- Tail recursion generally requires extra "accumulator" arguments to pass partial results
  - May require an auxiliary function

Tail Recursion - length

- How can we write length with tail recursion?

```
let length list =
    let rec length_aux list acc_length =
          match list
           with [ ] -> acc_length
             | (x::xs) ->
               length_aux xs (1 + acc_length)
     in length_aux list  0
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
  let rec num_neg_aux list curr_neg =
```

```
   in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] ->
      | (x :: xs) ->




  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
   match list with [] -> curr_neg
     | (x :: xs) ->




 in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
        num_neg_aux xs  ?



 in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
         num_neg_aux xs
           (if x < 0 then 1 + curr_neg
            else curr_neg)
  in num_neg_aux  ?  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
    match list with [] -> curr_neg
      | (x :: xs) ->
         num_neg_aux xs
           (if x < 0 then 1 + curr_neg
            else curr_neg)
  in num_neg_aux list  ?
```

Your turn: num_neg – tail recursive

```
# let num_neg list =
 let rec num_neg_aux list curr_neg =
   match list with [] -> curr_neg
     | (x :: xs) ->
       num_neg_aux xs
         (if x < 0 then 1 + curr_neg
          else curr_neg)
 in num_neg_aux list 0
```

```
let num_neg list =
List.fold_left
  (fun curr_neg -> (fun x ->
     (if x < 0 then 1 + curr_neg else curr_neg)
                    )
  )
0
list
```

Folding

```
# let rec fold_left f a list = match list
  with [] -> a | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
  <fun>
```

$\boxed{\text{fold\_left f a } [x_1;\ x_2;\ldots;x_n] = \text{f}(\ldots(\text{f (f a } x_1)\ x_2)\ldots)x_n}$

```
# let rec fold_right f list b = match list
  with [ ] -> b | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
  <fun>
```

$\boxed{\text{fold\_right f } [x_1;\ x_2;\ldots;x_n]\ \text{b} = \text{f}\ x_1(\text{f}\ x_2\ (\ldots(\text{f}\ x_n\ \text{b})\ldots))}$

Folding

- Can replace recursion by fold_right in any forward primitive recursive definition
  - Primitive recursive means it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by fold_left in any tail primitive recursive definition

Mapping Recursion

```
# let rec map f list =
  match list
  with [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]
# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
```

Map is forward recursive

```
# let rec map f list =
  match list
  with [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
# let map f list =
    List.fold_right (fun h -> fun r -> (f h) :: r)
                    list [];;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
```

Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```
# let doubleList list =
    List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```
# let doubleList list =
    List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
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```

- Same function, but no explicit recursion