Programming Languages and Compilers (CS 421)

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https://courses.engr.illinois.edu/cs421/sp2023

Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha
Background for Unification

- Terms made from constructors and variables (for the simple first order case)
- Constructors may be applied to arguments (other terms) to make new terms
- Variables and constructors with no arguments are base cases
- Constructors applied to different number of arguments (arity) considered different
- Substitution of terms for variables
Unification Problem

Given a set of pairs of terms ("equations")
\[\{(s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n)\}\]
(the **unification problem**) does there exist a substitution \(\sigma\) (the **unification solution**) of terms for variables such that
\[\sigma(s_i) = \sigma(t_i),\]
for all \(i = 1, \ldots, n\)?
Unification Algorithm

Let $S = \{(s_1 = t_1), (s_2 = t_2), \ldots, (s_n = t_n)\}$ be a unification problem.

Case $S = \{ \}$: $\text{Unif}(S) = \text{Identity function}$ (i.e., no substitution)

Case $S = \{(s, t)\} \cup S'$: Four main steps
Unification Algorithm

- **Delete:** if \( s = t \) (they are the same term) then \( \text{Unif}(S) = \text{Unif}(S') \)

- **Decompose:** if \( s = f(q_1, \ldots, q_m) \) and \( t = g(r_1, \ldots, r_n) \) if \( f = g \), \( m = n \), then
  \[
  \text{Unif}(S) = \text{Unif}(\{(q_1, r_1), \ldots, (q_m, r_m)\} \cup S')
  \]
  else **fail**!

- **Orient:** if \( t = x \) is a variable, and \( s \) is not a variable, \( \text{Unif}(S) = \text{Unif}(\{(x = s)\} \cup S') \)
Unification Algorithm

- **Eliminate:** if \( s = x \) is a variable, then if \( x \) does not occur in \( t \) (the occurs check), then
  - Let \( \varphi = \{x \rightarrow t\} \)
    - \( \text{Unif}(S) = \text{Unif}(\varphi(S')) \circ \{x \rightarrow t\} \)
  - Let \( \psi = \text{Unif}(\varphi(S')) \)
  - \( \text{Unif}(S) = \{x \rightarrow \psi(t)\} \circ \psi \)
    - Note: \( \{x \rightarrow a\} \circ \{y \rightarrow b\} = \{y \rightarrow ((\{x \rightarrow a\}(b))) \circ \{x \rightarrow a\} \) if \( y \) not in \( a \)
      else **fail** (because of occurs check failure)
Example

- $x, y, z$ variables, $f, g$ constructors

- Unify $\{(f(x) = f(g(f(z), y))), (g(y, y) = x)\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- $S = \{(f(x) = f(g(f(z), y))), (g(y, y) = x)\}$ is nonempty

Unify $\{(f(x) = f(g(f(z), y))), (g(y, y) = x)\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- Pick a pair: $(g(y, y) = x)$

- Unify \{ $(f(x) = f(g(f(z), y)))$, $(g(y, y) = x)$ \} = ?
Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(g(y,y)) = x$
- Orient: $(x = g(y,y))$

Unify \{(f(x) = f(g(f(z),y))), (g(y,y) = x)\} = \{(f(x) = f(g(f(z),y))), (x = g(y,y))\} \text{ by Orient}
Example

- $x, y, z$ variables, $f, g$ constructors

- Unify $\{(f(x) = f(g(f(z), y))), (x = g(y, y))\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- $\{(f(x) = f(g(f(z), y))), (x = g(y, y))\}$ is non-empty

- Unify $\{(f(x) = f(g(f(z), y))), (x = g(y, y))\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- Pick a pair: $(x = g(y, y))$

Unify $\{ (f(x) = f(g(f(z), y))), (x = g(y, y)) \} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- Pick a pair: $(x = g(y, y))$
- Eliminate $x$ with substitution $\{x \mapsto g(y, y)\}$
  - Check: $x$ not in $g(y, y)$
- Unify $\{(f(x) = f(g(f(z), y))), (x = g(y, y))\}$ = ?
Example

- **x, y, z** variables, **f, g** constructors
- Pick a pair: \((x = g(y,y))\)
- Eliminate \(x\) with substitution \(\{x\rightarrow g(y,y)\}\)

- Unify \(\{(f(x) = f(g(f(z),y))), (x = g(y,y))\} = \) Unify \(\{(f(g(y,y)) = f(g(f(z),y))))\}
  \(\circ \{x\rightarrow g(y,y)\}\)
Example

- $x, y, z$ variables, $f, g$ constructors

- Unify \{\((f(g(y,y)) = f(g(f(z),y)))\)\}
  
  $o \{x \mapsto g(y,y)\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- $\{(f(g(y,y)) = f(g(f(z),y)))\}$ is non-empty

- Unify $\{(f(g(y,y)) = f(g(f(z),y)))\} \circ \{x \mapsto g(y,y)\} = ?$
Example

- x, y, z variables, f, g constructors
- Pick a pair: \((f(g(y,y)) = f(g(f(z),y)))\)

- Unify \{\((f(g(y,y)) = f(g(f(z),y)))\}\)
  
  o \(\{x \mapsto g(y,y)\} = ?\)
Example

- $x, y, z$ variables, $f, g$ constructors
- Pick a pair: $(f(g(y, y)) = f(g(f(z), y)))$
- Decompose: $(f(g(y, y)) = f(g(f(z), y)))$
  becomes $\{(g(y, y) = g(f(z), y))\}$

- Unify $\{(f(g(y, y)) = f(g(f(z), y)))\}$
  $\circ \{x \rightarrow g(y, y)\} =$
- Unify $\{(g(y, y) = g(f(z), y))\} \circ \{x \rightarrow g(y, y)\}$
Example

- $x,y,z$ variables, $f,g$ constructors
- $\{(g(y,y) = g(f(z),y))\}$ is non-empty

- Unify $\{(g(y,y) = g(f(z),y))\} \circ \{x \mapsto g(y,y)\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- Pick a pair: $(g(y, y) = g(f(z), y))$

Unify \( \{(g(y, y) = g(f(z), y))\} \)

\[ \{x \mapsto g(y, y)\} = ? \]
Example

- \(x,y,z\) variables, \(f,g\) constructors
- Pick a pair: \((f(g(y,y)) = f(g(f(z),y)))\)
- Decompose: \((g(y,y)) = g(f(z),y))\) becomes \{\((y = f(z)); (y = y)\)\}

- Unify \{\((g(y,y) = g(f(z),y))) \ o \{x \rightarrow g(y,y)\}\} = \text{Unify} \{\((y = f(z)); (y = y)\) \ o \{x \rightarrow g(y,y)\}\}
Example

- $x, y, z$ variables, $f, g$ constructors

- Unify $\{(y = f(z)); (y = y)\} \circ \{x \mapsto g(y,y)\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- $\{(y = f(z)); (y = y)\} \circ \{x \mapsto g(y, y)\}$ is non-empty
- Unify $\{(y = f(z)); (y = y)\} \circ \{x \mapsto g(y, y)\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- Pick a pair: $(y = f(z))$

- Unify $\{(y = f(z)); (y = y)\} \circ \{x \mapsto g(y, y)\} = ?$
Example

- \(x, y, z\) variables, \(f, g\) constructors
- Pick a pair: \((y = f(z))\)
- Eliminate \(y\) with \(\{y \rightarrow f(z)\}\)
- Unify \(\{(y = f(z)); (y = y)\} \circ \{x \rightarrow g(y, y)\} = \text{Unify} \{(f(z) = f(z))\}\)
  - \(\circ (\{y \rightarrow f(z)\} \circ \{x \rightarrow g(y, y)\}) = \text{Unify} \{(f(z) = f(z))\}\)
  - \(\circ \{y \rightarrow f(z); x \rightarrow g(f(z), f(z))\}\)
Example

- $x, y, z$ variables, $f, g$ constructors

- Unify $\{(f(z) = f(z))\}$
  
  $\circ \{y \rightarrow f(z); x \rightarrow g(f(z), f(z))\} = ?$
Example

- \( x, y, z \) variables, \( f, g \) constructors
- \( \{ (f(z) = f(z)) \} \) is non-empty

- Unify \( \{ (f(z) = f(z)) \} \)
  \[ \circ \{ y \rightarrow f(z); x \rightarrow g(f(z), f(z)) \} = ? \]
Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(f(z) = f(z))$

- Unify $\{(f(z) = f(z))\}$
  - $\circ \{y \rightarrow f(z); x \rightarrow g(f(z), f(z))\} = ?$
Example

- $x,y,z$ variables, $f,g$ constructors
- Pick a pair: $(f(z) = f(z))$
- Delete
- Unify $\{(f(z) = f(z))\}$
- $\circ \{y \rightarrow f(z); x \rightarrow g(f(z), f(z))\} =$
- Unify $\{} \circ \{y \rightarrow f(z); x \rightarrow g(f(z), f(z))\}$
Example

- $x, y, z$ variables, $f, g$ constructors

- Unify $\{\} \circ \{y \rightarrow f(z); x \rightarrow g(f(z), f(z))\} = ?$
Example

- $x, y, z$ variables, $f, g$ constructors
- $\{\}$ is empty
- Unify $\{\} = \text{identity function}$
- Unify $\{\} \circ \{y \rightarrow f(z); x \rightarrow g(f(z), f(z))\} = \{y \rightarrow f(z); x \rightarrow g(f(z), f(z))\}$
Example

Unify \{ (f(x) = f(g(f(z),y))), (g(y,y) = x) \} = \{ y \rightarrow f(z); x \rightarrow g(f(z), f(z)) \}

\[
f(x) = f(g(f(z), y))
\]
\[
\rightarrow f(g(f(z), f(z))) = f(g(f(z), f(z)))
\]

\[
g(y, y) = x
\]
\[
\rightarrow g(f(z), f(z)) = g(f(z), f(z))
\]
Example of Failure: Decompose

Unify\{ (f(x, g(y)) = f(h(y), x)) \}

Decompose: \( f(x, g(y)) = f(h(y), x) \)

= Unify \{ (x = h(y)), (g(y) = x) \}

Orient: \( g(y) = x \)

= Unify \{ (x = h(y)), (x = g(y)) \}

Eliminate: \( x = h(y) \)

Unify \{ (h(y) = g(y)) \} o \{ x \rightarrow h(y) \}

Decompose only rule in this case, but Decompose fails!
Example of Failure: Occurs Check

- Unify\{((f(x,g(x)) = f(h(x),x))}\}
- Decompose: \((f(x,g(x)) = f(h(x),x))\)
- \(= \) Unify \{\((x = h(x)), (g(x) = x)\)\}
- Orient: \((g(x) = x)\)
- \(= \) Unify \{\((x = h(x)), (x = g(x))\)\}
- Eliminate only rule that applies in this case, but Eliminate fails because the occurs check fails.